Physics 606: Homework #5

Due April 17, 2018

Jackson: Problems 6.12, 6.14, 7.14, 7.16 not graded

1) A magnetic field \( \mathbf{B}(r,\phi,z) = e_z \, g(t) \, r^2 \) exists in all space.

   a) Find an electric field consistent with this \( \mathbf{B} \).

   b) Find the energy density in fields and the Poynting flux.

   c) Based on the above, discuss energy conservation.

2) A vacuum gap of thickness \( \Delta \) exists between two slabs of material of dielectric constant \( \varepsilon_3 > \varepsilon_1 > \varepsilon_0 \). Radiation, with frequency \( \omega \), is incident from the slab on the left (see below) at an angle \( \theta_1 \) to the normal. Take \( \mu = \mu_0 \).

   a) Give expressions for the angles of the plane waves in each of the three regions.

   b) Assume the incident wave is TE polarized with respect to the \( z \)-axis. Write expressions for the wave fields with yet to be determined amplitudes in each region.

   c) Express the constraints on the amplitudes in each region that are imposed by the boundary conditions at each surface.

   d) Calculate the effective reflection coefficient seen at the boundary between regions 1 and 2. (The fact that there are two boundaries is important. Introduce intermediate variables to simplify the expressions, an effective impedance at the surface between regions 1 and 2 would be helpful.)

   e) What is the critical angle for total internal reflection in the limit \( \Delta \to \infty \)?

   f) Suppose the angle of incidence is above that for total internal reflection, estimate the amount of transmitted power for large values of \( \Delta \).
3. The Smith Island Cake is the official dessert of the State of Maryland. It consists of alternating layers of two dielectric materials as pictured at right. Suppose the dielectric constants and the thicknesses of the two alternating layers are $\varepsilon_1, \varepsilon_2$ and $d_1, d_2$, respectively. A conscientious and thorough baker (Chef L. Brillouin, a graduate of a French culinary school) wishes to characterize his cake. First he considers the effective dielectric tensor of his cake at long wavelength. By effective dielectric tensor, he means the approximate tensor that applies to fields with scale length greater than the thickness of the layers.

A) What are the nine elements of this tensor? Take the $z$-axis to point normal to the planes of the layers. Recall that if $\mathbf{P}$ is the polarization density and $\mathbf{E}$ the electric field the tensor is such that $\varepsilon \cdot \mathbf{E} = \mathbf{P} + \varepsilon_0 \mathbf{E}$.

B) Realizing that characterising the cake at long wavelength may not be sufficient, the baker considers the propagation of waves through the layers of the cake. To make his life simple he considers waves that propagate in the $z$ direction, that is normal to the interface surfaces, so that polarization is unimportant. Let $E(0)$ and $H(0)$ be the amplitudes of the electric and magnetic fields at $z=0$, the boundary of one layer of the cake. Show that at $z=d$ the other boundary of that layer the values of $E(d)$ and $H(d)$ are given by

$$
\begin{pmatrix}
E(d) \\
H(d)
\end{pmatrix} =
\begin{bmatrix}
\cos \theta & iZ \sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{pmatrix}
E(0) \\
H(0)
\end{pmatrix}.
$$
where $Z = \sqrt{\mu_0 / \varepsilon}$ is the impedance of the material in that layer, and

$\theta = kd = \omega \sqrt{\varepsilon \mu_0} d$.

C) Now consider propagation through an infinite sequence of alternating layers of thickness $d_1 + d_2$. Show that solutions for $E$ and $H$ are of the form

$$
\begin{pmatrix}
E(z + d_1 + d_2) \\
H(z + d_1 + d_2)
\end{pmatrix} = \lambda
\begin{pmatrix}
E(z) \\
H(z)
\end{pmatrix}.
$$

Obtain an equation for $\lambda$ (there are two values whose product is unity). Do not get too bogged down in algebra, just indicate how $\lambda$ is determined.

D) Since the product of the two lamda value is unity, interpret what this means when lamda is real and when lamda is complex.