Physics 606: Homework #1

Due: Tuesday Feb 6, 2018

Jackson: Problems 1.1, 1.6, 1.12, and 1.13 will not be graded

**Prob. A)** A volume charge density \( \rho(x) = \rho_0 + \rho_1(x/d) \), lies between two conducting planes at \( x=0 \) and \( x=d \). Here \( \rho_0 \) and \( \rho_1 \) are given constants. The potential at the plane \( x=0 \) is held at 0, and the potential at the plane \( x=d \) is held at \( V \) volts.

a) Find the potential as a function of \( x \) for \( 0<x<d \).

b) Suppose you were to solve this problem by a variational principle. What would be a good choice for a trial function?

**Prob B)** The potential surrounding a point charge, \( q \), located at position \( x_0 \) in a plasma is given by,

\[
\Phi_0(x) = \frac{q \exp(-|x-x_0|/\lambda_d)}{4\pi\varepsilon_0|x-x_0|},
\]

where \( \lambda_d \) is known as the Debye length. a) Find the charge density induced in the plasma. b) Show that it is proportional to the local potential.

**Prob C)** An unkown charge density \( \rho(x) \) lies between two square, large, parallel conducting plates of area \( A \) and separated by distance \( d \ll A^{1/2} \). Initially the plates are uncharged and are isolated. The plates are then grounded and a charge \( Q_1 \) escapes from one plate (located in the plane \( x_1 \)) and \( Q_2 \) escapes from the other plate (located in the plane \( x_2 \)). Take \( x_2 > x_1 \). Based on this information, how much total charge is in \( \rho(x) \). What is the \( x \)-component of the dipole moment of the charge distribution,

\[
p_x = \int d^3x \: x \rho(x) ?
\]

Read "The Green on Green Functions" in the HNDO directory of the course web site.
Problem A: \[ \frac{d^2 \phi}{dx^2} = -\left( \frac{p_0 + p_1 x}{E_0} \right) \]
\[ \begin{align*}
\phi(0) &= 0 \\
\phi(d) &= V
\end{align*} \]

A) Since the RHS is in the form of a polynomial in \( x \), try this \( \phi(x) = Ax + Bx^2 + Cx^3 + Dx^4 \).

Equivalently one can integrate \( \phi(x) \) as well.

Note \( A = 0 \) to satisfy \( \phi(0) = 0 \).

\[ \begin{align*}
\frac{d^2 \phi}{dx^2} &= 2C + 6DX = -\left( \frac{p_0 + p_1 x}{E_0} \right) \\
C &= -\frac{p_0}{2E_0} \\
D &= -\frac{p_1}{6E_0}
\end{align*} \]

\[ \phi(x) = Bx - \frac{p_0 x^2}{2E_0} - \frac{p_1 x^3}{6E_0} \]

Now satisfy \( \phi(d) = V \) and \( V = Bd - \left( \frac{p_0}{2E_0} + \frac{p_1}{6E_0} \right) d^2 \).

\[ B = \frac{V + \left( \frac{p_0}{2E_0} + \frac{p_1}{6E_0} \right) d}{d} \]

\[ \phi(x) = \frac{x}{d} V + \left( \frac{p_0 + p_1}{2E_0 + 6E_0} \right) x d \]

\[ \frac{p_0^2}{2E} - \frac{p_1}{6E_0} x^3 \]

B) Try polynomial \( \phi(x) = Ax + Bx^2 + Cx^3 + Dx^4 \).

must satisfy \( \) boundary conditions \( \phi(0) = 0 \Rightarrow A = 0 \)

\[ \phi(d) = V = Bd + Cd^2 + Dd^3 \]

Thus \( \phi(x) = \frac{x}{d} \left( V - Cd^2 - Dd^3 \right) + Cx^2 + Dx^3 \)

\( V = C \) and \( C = D \).
\[ \phi(x) = \frac{V}{d} x^2 + C(x^2 - x) + D(x^2 - x) \]

\[ \frac{d\phi}{dx} = \frac{V}{d} + C(2x - d) + D(3x^2 - d) \]

\[ I_1 = \frac{1}{2} \int_0^d dx \left( \frac{d\phi}{dx} \right)^2 \]

\[ I = I_1 - I_2 \]

\[ I_2 = \frac{1}{2} \int_0^d dx \phi \left[ \rho_0 + \frac{\rho_1}{2} \right] \left( x^2 - x \right) \]

\[ \frac{\partial I_1}{\partial C} = \int_0^d dx \left[ \frac{V}{d} + C(2x - d) + D(3x^2 - d) \right] (2x - d) \]

\[ \frac{\partial I_2}{\partial C} = 0 \cdot \frac{V}{d} + \frac{d^4}{3} C + \frac{1}{2} \cdot d^4 D \]

\[ I_2 = \int_0^d dx \left( \rho_0 + \frac{\rho_1}{2} \right) \left( \frac{V}{d} x + C(x^2 - x) + D(x^2 - x) \right) \]

\[ \frac{\partial I_2}{\partial C} = \frac{1}{60} \int_0^d dx \left( \rho_0 + \frac{\rho_1}{2} \right) (x^2 - x) = -\frac{d^5}{60} \left[ \rho_0 + \rho_1 \right] \]

\[ \frac{\partial I_1}{\partial D} = \int_0^d dx \left[ \frac{V}{d} + C(2x - d) + D(3x^2 - d) \right] (3x^2 - x) \]

\[ \frac{\partial I_1}{\partial D} = 0 \cdot \frac{V}{d} + C \cdot \frac{d^4}{2} + D \cdot \frac{d^5}{5} \]

\[ \frac{\partial I_2}{\partial D} = \frac{1}{60} \int_0^d dx \left( \rho_0 + \frac{\rho_1}{2} \right) \left( x^3 - x^2 \right) \]
\[ \frac{\partial I^2}{\partial D} = -\frac{d^4}{e_0} \left[ \frac{p_0}{4} + \frac{2p_1}{15} \right] \]

\[ \exists \frac{\partial I}{\partial C} = \delta (I_1 - I_2) = 0 = d^3 \left( \frac{C}{3} + \frac{dD_A}{2} \right) + d^3 \left( \frac{p_0 + p_1}{6} \right) \]

\[ \frac{\partial I}{\partial D} = \exists (I_1 - I_2) = 0 = d^4 \left( \frac{C}{2} + \frac{dD_A}{5} \right) + d^4 \left( \frac{P_0}{4} + \frac{2P_1}{15} \right) \]

Solving for \( C \) gives \( C = -\frac{P_0}{2e_0} \), \( D = -\frac{P_1}{6e_0} \).
Problem 48  In Cartesian.

In plasma, the potential surrounding a point charge \( q \) is

\[
\phi_0 (\mathbf{x}) = \frac{q}{4\pi \epsilon_0} \frac{-e^{r/d}}{|x-x_0|}
\]

Simplified:

\[
\phi_0 (r) = \frac{q}{4\pi \epsilon_0} \frac{-e^{r/d}}{r}, \text{ due to symmetry,}
\]

Poisson's equation: \( f(r) = -\epsilon_0 \nabla^2 \phi \)

\( \nabla^2 \rightarrow \frac{d^2}{dr^2} \)

\[
\frac{d\phi_0}{dr} = \frac{-e^{r/d}}{4\pi \epsilon_0} \left( \frac{1}{r^2} + \frac{1}{r d} \right)
\]

\[
\frac{d^2 \phi_0}{dr^2} = \frac{-e^{r/d}}{4\pi \epsilon_0} \left\{ \left( \frac{1}{r^2} + \frac{1}{r d} \right) \frac{1}{r d} + \frac{2}{r^3} + \frac{1}{r^2 d} \right\}
\]

\( \Rightarrow \)

\[
\phi_0 (r) = \frac{-e^{r/d}}{4\pi \epsilon_0} \frac{1}{r} \left\{ \frac{2}{r d} + \frac{1}{r d} + \frac{2}{r^2} \right\}
\]

Examine

\[
\frac{2}{r^2} + \frac{2}{r d d} + \frac{1}{r^2} = \frac{2 r d + 2 r d + r^2}{r^2 d^2}
\]
\[ = \frac{2 \left( \frac{2d}{r} \right)^2 + 2 \frac{2d}{r} + 1}{4d^2} \]

If \( r \gg 4d \) then \( \frac{2d}{r} \to 0 \)

\[ \Rightarrow \phi(r) \approx -\frac{q}{4\pi \varepsilon_0} \frac{e^{-r/4d}}{r} = -\phi_0 \frac{e_0}{4d} \]

\( \Rightarrow \) charge distribution is proportional to potential.
\[
\begin{align*}
\text{Solve} \quad & \nabla^2 \phi = -\frac{\rho(x_1)}{\varepsilon_0} \quad \text{P.E.} \\
\phi(x_1) = \phi(x_2) = 0 \\
\end{align*}
\]

a) Multiply P.E. by \((x-x_1)\) and integrate \(d^3x\) volume between plates.

\[
\int_V d^3x (x-x_1) \nabla^2 \phi = -\frac{1}{\varepsilon_0} \int_V \nabla \cdot (x-x_1) \rho(x) = \int_V \left( x-x_1 \left[ \frac{d\phi}{dx} \right]_1 \right) - \phi \left[ \frac{d\phi}{dx} \right]_2 \right) \}
\]

\[
-\frac{1}{\varepsilon_0} \int_V d^3x (x-x_1) \rho(x) = (x_2-x_1) \left[ \frac{d\phi}{dx} \right]_2 = (x_2-x_1) \frac{(Q_2)}{\varepsilon_0}
\]

\[Q_2 = \int_V d^3x \frac{(x-x_1)}{(x_2-x_1)} \rho(x) \]

Similarly \(Q_1 = \int_V d^3x \frac{x-x_2}{x_1-x_1} \rho(x) = \int_V d^3x \frac{x-x_2}{x_2-x_1} \rho(x)\)

Add

\[(Q_1 + Q_2) = \int_V d^3x \rho(x) \]

Subtract

\[(x_2-x_1)(Q_2-Q_1) = \int_V d^3x (2x-(x_1+x_2)) \rho(x)\]
Thus

\[ \int d^3x \rho(x) = \frac{1}{2} \left\{ (X_2 - x_1)(Q_2 - Q_1) + (Q_2 + Q_1)(x_1 + x_2) \right\} \]
Jackson 1.1

1.1(a)

By definition, charges are free to move inside a conductor in response to an electric field. Choose an arbitrary Gaussian surface entirely inside conductor and apply Gauss’ law,

$$\oint \mathbf{E} \cdot d\mathbf{a} = Q_{\text{enc}}/\varepsilon_0$$

In equilibrium, charges are not moving inside the conductor so $\mathbf{E} = 0$, which implies $Q_{\text{enc}} = 0$. However, the chosen surface is immaterial to this reasoning. Thus we can choose a surface that encompasses the entire conductor except the surface. The enclosed charge in this volume is zero from above, so therefore all added charge must be on the surface.

1.1(b)

Consider first the case of enclosed charge. The conductor has charge $Q$ and the enclosed charge is $q$. Using a surface just outside the conductor, Gauss’ law gives

$$\oint \mathbf{E} \cdot d\mathbf{a} = Q_{\text{enc}}/\varepsilon_0$$

$$E_{\text{perp}} A = (Q + q)/\varepsilon_0$$

$$E_{\text{perp}} = (Q + q)/\varepsilon_0 A$$

As is evident, $E_{\text{perp}}$ is non-zero just outside the conductor, so the exterior is not shielded from interior charges.

Consider now the opposite case of charge exterior to the conductor. We construct a closed path connecting two points $A$ and $B$ on the surface of the conductor: one leg of the path travels along the surface of the conductor, and the return path passes directly through the cavity. By (1.21),

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\int_B^A \mathbf{E}_{\text{surface}} \cdot d\mathbf{l}_{\text{surface}} + \int_A^B \mathbf{E}_{\text{cavity}} \cdot d\mathbf{l}_{\text{cavity}} = 0$$

$$\int_B^A \mathbf{E}_{\text{surface}} \cdot d\mathbf{l}_{\text{surface}} = -\int_A^B \mathbf{E}_{\text{cavity}} \cdot d\mathbf{l}_{\text{cavity}}$$

However, again, charges move freely in a conductor in response to a field, so in equilibrium $E_{\text{surface}} = 0$. Thus, the path integral through the cavity is zero, and $E_{\text{cavity}} = 0$. 

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1.1(c)

We know that in equilibrium, the internal electric field of a conductor is zero. Therefore, on the boundary, the surface electric field parallel to the boundary is zero (otherwise charges would move around on the surface of the conductor). At a boundary, $\nabla \times (\vec{E}_1 - \vec{E}_2) = 0$, so the perpendicular component of the electric field is continuous. Since the internal perpendicular field is zero and continuous across the boundary, the external electric field must be perpendicular to the conductor. Consider a pillbox on the surface (small enough to ignore curvature) of area $A$. By Gauss' law,

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$\oint_{ext} \vec{E}_{ext} \cdot d\vec{a} + \oint_{int} \vec{E}_{int} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$E_{ext}A + E_{int}A = \frac{\sigma A}{\epsilon_0}$$

Again, $E_{int} = 0$, so $E_{ext} = \frac{\sigma}{\epsilon_0}$. 

Figure 1: Illustration of the path used in solution of problem 1.1(b)
Jackson 1.6

1.6(a)

For a large conducting sheet with charge density \( \sigma \), \( \vec{E} = \sigma \hat{n} / \varepsilon_0 = q \hat{n} / A \varepsilon_0 \) (ignoring edge effects). Thus, the voltage difference between two sheets separated by \( d \) is

\[
V = \oint \vec{E} \cdot d\vec{l} = \frac{qd}{A\varepsilon_0}
\]

calculated along a path perpendicular to the sheets. The capacitance is thus

\[
C = \frac{q}{V} = \frac{A \varepsilon_0}{d}.
\]

Jackson 1.12

We compare the given geometry to a parallel plate capacitor with applied voltage \( \Phi'_0 \) (and charge \( q' \) on the plates) and no charge between the plates. We use our knowledge of this system in conjunction with Green's reciprocation to solve the problem:

\[
\begin{align*}
\rho &= q\delta(\vec{a} - \vec{x}), \rho' = 0 \\
\Phi(0) &= \Phi(d) = 0, \Phi'(x) = \Phi'_0 x / d \\
\sigma &= \frac{Q}{A}, \sigma' = \frac{q'}{A}
\end{align*}
\]

Plugging in, we get

\[
\begin{align*}
\int_V q\delta(\vec{a} - \vec{x}) \frac{\Phi'_0 x}{d} d^3x + \oint_A (0 + \Phi'_0) \frac{-q}{A} dS \\
&= \int_V 0 \cdot \Phi dV + \oint_A \frac{q'}{A} \cdot 0 dS \\
&= q\Phi'_0 \frac{a}{d} + \Phi'_0 Q = 0
\end{align*}
\]

where the sum in the surface integral corresponds to the bottom and top plates, respectively. Thus, \( Q = -qa/d \), as expected.
\[
\phi(1) = 0 = \phi \in S_p \quad \text{and} \quad \phi = \int \phi \frac{dS_p}{\sqrt{g}} + \int \frac{\sqrt{g}}{\lambda^2} \phi \cdot \phi = 0
\]

\[
\phi = \phi(1) = 0 \quad \text{and} \quad \phi = \int \phi \frac{dS_p}{\sqrt{g}} + \int \frac{\sqrt{g}}{\lambda^2} \phi \cdot \phi = 0
\]

\[
\phi = \phi(1) = 0 \quad \text{and} \quad \phi = \int \phi \frac{dS_p}{\sqrt{g}} + \int \frac{\sqrt{g}}{\lambda^2} \phi \cdot \phi = 0
\]