1. Force acting on charge 1 due to charge 2 (Coulomb’s Law)

\[ F_i = \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j (x_i - x_j)}{|x_i - x_j|^3} \]

2. Force on charge 1 due to many charges, 2-n (Superposition)

\[ F_i = \frac{1}{4\pi\varepsilon_0} \sum_{j \neq 1} \frac{q_i q_j (x_i - x_j)}{|x_i - x_j|^3} \]

3. Electrostatic field at \( x_i \) due to charges 2-n (factor out \( q_1 \))

\[ F_i = q_i E(x_i) \]
\[ E(x_i) = \frac{1}{4\pi\varepsilon_0} \sum_{j \neq 1} \frac{q_j (x_i - x_j)}{|x_i - x_j|^3} \]

Must exclude contribution from self-field when dealing with point charges.

4. Field due to a continuous charge density \( \rho(x) \),

\[ E(x) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(x')(x - x')}{|x - x'|^3} \]

For continuous (finite) charge density no need to exclude \( x=x' \).

5. Example, spherically symmetric charge density

6. Coulomb’s Law implies Gauss’ Law

\[ \varepsilon_0 \int_S \mathbf{n} \cdot \mathbf{E} = \int_V \varepsilon_0 \rho(x) = \int_V \rho(x) = q_{\text{enclosed}} \]

\( S \) is any closed surface and \( V \) is the enclosed volume. \( \mathbf{n} \) is the outward normal.
7. Coulomb’s Law implies a scalar potential

\[ E(x) = -\nabla \Phi(x) \]

\[ \oint E(x) \cdot dl = 0 \]

\[ \Phi(x) = \frac{1}{4\pi \varepsilon_0} \int d^3x' \frac{\rho(x')}{|x - x'|} + \text{const.} \]

Must include all charge in integral. If const. = 0, \( \Phi(x) \to 0 \) as \( |x| \to \infty \).

8. Gauss’ Law applied locally is Poisson’s equation

\[ \varepsilon_0 \nabla \cdot E(x) = \rho(x) \]

Must include conservative property to determine field.

\[ E(x) = -\nabla \Phi(x) \]

9. Boundary conditions at a surface separating region 1 from region 2. Normal \( n \) points from 1 to 2.

Tangential components of electric field are continuous,

\[ n \times (E_2 - E_1) = 0, \]

Normal component jumps discontinuously in the presence of a surface charge density \( \sigma(x) \)

\[ n \cdot (E_2 - E_1) = \sigma(x) / \varepsilon_0 \]

While both of the above are true, they have very different application when solving for unknown fields. You almost always apply continuity of the tangential fields to determine the electric field. Once the field has been found you examine the jump in the normal component to find the surface charge density. The situation is different when regions 1 and 2 have different dielectric constants.

10. Solutions are unique. If you find a solution for the potential that satisfies the equations listed in (8) and the boundary conditions (Dirichlet of Neumann) that is the solution, there are no others.

11. Green’s theorem
\[
\int_V d^3x \left( \Psi \nabla^2 \Phi - \Phi \nabla^2 \Psi \right) = \int_S da \mathbf{n} \cdot \left( \Psi \nabla \Phi - \Phi \nabla \Psi \right)
\]

for any potentials \( \Psi, \Phi \).

12. Green functions,

Problem to be solved,

\[-\varepsilon_0 \nabla^2 \Phi(x) = \rho(x)\]

+ Boundary conditions

Green function \( \Psi = G \) satisfies

\[-\nabla^2 G(x,x') = 4\pi \delta(x - x')\]

Green’s theorem gives

\[
\Phi(x') = \int_V d^3x \frac{\rho(x)}{4\pi \varepsilon_0} G(x,x') + \frac{1}{4\pi} \int_S da \mathbf{n} \cdot (G\nabla \Phi - \Phi G).
\]

Pick BC’s on \( G \) to eliminate unknown terms in surface integral.

13. Electrostatic energy

Energy required to assemble a system of point charges, bringing each in from infinity,

\[
W = \frac{1}{8\pi \varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|x_i - x_j|}.
\]

\( i=j \) is excluded.

For continuous distributions of charge,

\[
W = \frac{1}{8\pi \varepsilon_0} \iint d^3x d^3x' \frac{\rho(x)\rho(x')}{|x - x'|} = \frac{1}{2} \int d^3x \rho(x)\Phi(x) = \frac{\varepsilon_0}{2} \int d^3x |E(x)|^2.
\]

For finite charge distributions \( x=x' \) is not a problem. The first two integrals are over regions containing charge, the last integral is over all volume.

14. Capacitance
For \( n \) conductors each held at a different potential \( V_i \), \( i=1,n \) and each with a different net charge \( Q_j \), \( j=1,n \). The charges and voltages are related linearly by the symmetric capacitance matrix \( C_{ji} \),

\[
Q_j = \sum_{i=1}^{n} C_{ji} V_i .
\]

Energy required to assemble charges,

\[
W = \frac{1}{2} \sum_{j=1}^{n} V_j Q_j = \frac{1}{2} \sum_{j=1}^{n} V_j C_{ji} V_i > 0
\]

15. Variational Principle

Minimizing the functional \( I(\psi) \)

\[
I(\psi) = \int d^3 x \left( \frac{\varepsilon_0}{2} |\nabla \psi(x)|^2 - \rho(x) \psi(x) \right),
\]

wrt all functions \( \psi(x) \) satisfying prescribed boundary conditions is equivalent to solving the Poisson equation subject to the same BC’s.

16. Thomson’s theorem: Consider \( n \) surfaces in space, labeled \( i=1,n \). On each surface a charge \( Q_i \) is placed. The charge is distributed over each surface with surface charge density \( \sigma_i(x) \), where

\[
\int_{S_i} \sigma_i(x) = Q_i .
\]

Thm: If the surface charge density functions are chosen to minimize the electric field energy, the resulting potential will be constant on each surface.

To show this we want to minimize the quantity \( W \) above, which in the case of surface charges is written,

\[
W = \frac{1}{8\pi\varepsilon_0} \sum_{i,j} \iint da da' \frac{\sigma_i(x)\sigma_j(x')}{|x-x'|} .
\]

Let the surface charge density be written in terms of the minimizing charge density and a test perturbation, \( \sigma_i(x) = \sigma_{i0}(x) + \varepsilon_i \sigma_{it}(x) \). The test charge density must correspond to zero net charge on each surface,
\[ \int_{S_i} da \sigma_i(x) = 0. \]

Otherwise it is unconstrained. The condition \( \frac{dW}{d\varepsilon_k} = 0 \) then gives,

\[
0 = \frac{1}{8\pi\varepsilon_0} \left\{ \sum_j \iiint da da' \frac{\sigma_u(x)\sigma_j(x')}{|x-x'|} + \sum_i \iiint da da' \frac{\sigma_u(x)\sigma_i(x')}{|x-x'|} \right\}.
\]

Switching \( x \) and \( x' \) and changing the dummy index \( i \) to \( j \) in the second term reveals that it is the same as the first term. The above condition then becomes,

\[
0 = \frac{1}{4\pi\varepsilon_0} \sum_j \iiint da da' \frac{\sigma_u(x)\sigma_j(x')}{|x-x'|} = \int_{S_i} da \sigma_u(x)\Phi_0(x),
\]

where \( \Phi_0(x) \) is the minimizing potential. Taking into account that the test charge density is any function corresponding to zero net charge on each surface it is seen that the minimum is reached only when the minimizing potential is a constant on each surface,

\[
\Phi_0(x)|_{S_i} = V_i.
\]