Snow by coordinate transformation and use of the law of addition of velocities that the expressions for the charge and current densities for a point charge, transform as a four-vector.

\[ \rho = q \delta (\vec{x} - \vec{x}_i(t)) \]
\[ \vec{J} = q \vec{U}_i(t) \delta (\vec{x} - \vec{x}_i(t)) \]

Consider two frames, moving at a velocity \( \vec{V} \) relative to one another. The coordinate position in each frame, call them the unprimed and primed frames, we know transforms as a 4-vector via Lorentz transformation, where we can write:

\[
\begin{align*}
ct' &= \gamma (ct - \beta \cdot \vec{x}) \\
ct &= \gamma (ct' + \beta \cdot \vec{x}') \\
x_{ii}' &= \gamma (x_{ii} - \beta ct) \\
x_{ii} &= \gamma (x_{ii}' + \beta ct') \\
\vec{x}'_t &= \vec{x}_t \\
\vec{x}_t &= \vec{x}'_t
\end{align*}
\]

where \( \gamma = \frac{1}{\sqrt{1 - \frac{\beta^2}{c^2}}} \) and \( \beta = \frac{\vec{V}}{c} \); the \( ii \) and \( t \) parts refer to those \( ii \) and \( t \) to the velocity \( \vec{V} \).

* We wish to show that via a coordinate transform such as this, \((\rho, \vec{J})\) transforms as a 4-vector via Lorentz transformation.

**SO NOW**: Consider a point charge moving at a constant velocity \( \vec{U} \) in the unprimed frame, where \( \vec{U} \) has components \( U_{ii} \) and \( U_t \) (parallel; perpendicular) to the relative velocity of the two frames \( \vec{V} \).

This constant velocity should be given by:

\[ \vec{U} = \frac{\vec{x}}{t} \]

where \( U_{ii} = \frac{x_{ii}}{t} \) and \( U_t = \frac{x_t}{t} \).
We can use what we have thus far to generate a relationship between the unprimed and primed delta functions.

\[ \delta(x - x_i) = \frac{\delta(x - \hat{u}t)}{\gamma(1 - \hat{u} \cdot \hat{x})} \]

* Now use the 4-vector Lorentz transformation written previously to substitute into the above...

\[ = \frac{\delta}{\gamma} \left\{ \gamma(x' + \beta ct') - \gamma'(c + \beta \cdot x') \right\} \delta \left[ \frac{x' - \hat{u} \gamma (c + \beta \cdot x')}{c} \right] \]

**By definition,** \( \delta(f(x)) = \sum_{\text{zeros of } f(x)} \delta(x - x_i) \)

**IN THIS CASE:**

\[ f(x_{ii}) = \gamma(x' + \beta ct') - \gamma'(c + \beta \cdot x') \]

which is zero when

\[ x_{ii}' = \frac{\gamma'}{\gamma} (c + \beta \cdot x') - \beta ct' \]

**will extract II component**

**Noting that** \( \hat{p} \cdot \hat{x}' = \frac{\gamma}{\gamma'} \hat{x} \cdot x' = \frac{\gamma}{\gamma'} x_{ii} \)

\[ f'(x_{ii}) = \gamma - \gamma' \frac{\gamma}{\gamma'} x_{ii} \]

**SO THEREFORE:**

\[ \therefore \delta(x - x_i(\cdot)) = \frac{1}{\gamma(1 - \gamma' \gamma') \gamma(1 - \gamma' \gamma') \gamma(1 - \gamma' \gamma')} \delta \left( \frac{x_i}{\gamma} - \frac{\gamma'(ct + \beta \cdot x') - \beta ct'}{\gamma} \right) \delta \left( \frac{x_i}{\gamma} - \frac{\gamma'(ct + \beta \cdot x')}{\gamma} \right) \]

**BUT NOTE:** In the primed frame, the point charge's velocity will be \( \hat{U} = \frac{\gamma'}{\gamma} \hat{x} \), with \( U_{ii}' = x_{ii}' + \frac{\gamma'}{\gamma} \cdot \frac{\gamma'}{\gamma} U_{ii} \)

*And by the law of addition of velocities...

**From Jackson**

\[ \hat{U}' = \frac{U_{ii} \gamma}{\gamma'(1 - \gamma' \gamma') + \gamma'(1 - \gamma' \gamma') - \gamma} \]

\[ \hat{U}' = \frac{U_{ii} \gamma}{1 - \gamma' \gamma' \gamma} \]

\[ U_{ii}' = \frac{U_{ii} \gamma}{1 - \gamma' \gamma' \gamma} \]
So we can rewrite our delta function expression as...

\[
\delta(\vec{x} - \vec{x}_i(t)) = \frac{1}{\gamma(1 - \frac{\vec{u} \cdot \vec{v}}{c^2})} \delta \left\{ x'_{ii} - \left[ \frac{u_{ii}}{c} (\vec{c} + \frac{\vec{v}}{c} \cdot \vec{x}) - \frac{\vec{v} \cdot \vec{c}}{c^2} \right] t + \delta \left\{ \vec{x}'_{ii} - \left[ \frac{u_{ii}}{c} (\vec{c} + \frac{\vec{v}}{c} \cdot \vec{x}) \right] t \right\}
\]

\[
= \frac{1}{\gamma(1 - \frac{\vec{u} \cdot \vec{v}}{c^2})} \delta \left\{ x'_{ii} - \left[ \frac{u_{ii}}{c} (1 + \frac{\vec{v} \cdot \vec{u}}{c^2}) \right] t \right\} \delta \left\{ \vec{x}'_{ii} - \left[ \frac{u_{ii}}{c} (1 + \frac{\vec{v} \cdot \vec{u}}{c^2}) \right] t \right\} = \delta(\vec{x}' - \vec{x}_i(t))
\]

**NOTE:** \( u_{ii} = \vec{v} \cdot \vec{u} \) since this will pull out component of \( \vec{u} \parallel \vec{v} \)

**SO FINALLY:** \( \delta(\vec{x} - \vec{x}_i(t)) = \delta(\vec{x}'_{ii} - u_{ii} t) \delta(\vec{x}'_{ii} - u_{ii} t) = \delta(\vec{x}' - \vec{x}_i(t)) \)

\[
\gamma(1 - \frac{\vec{v} \cdot \vec{u}}{c^2}) \gamma(1 - \frac{\vec{v} \cdot \vec{u}}{c^2})
\]

**SO NOW:** If we Lorentz transform the presumed 4-vector (\( \rho, \vec{J} \)), we'd like to show the results found correspond to what we'd write for \( \rho' \) and \( \vec{J}' \) in the primed frame:

\[
\rho' = q \delta(\vec{x}' - \vec{x}_i(t))
\]

\[
\vec{J}' = q \vec{u}_i \delta(\vec{x}' - \vec{x}_i(t))
\]

**THE LORENTZ TRANSFORM of** (\( \rho, \vec{J} \))

\[
\begin{align*}
\rho' &= \gamma (\rho - \vec{p} \cdot \vec{J}) \\
u_{ii}' &= \gamma (u_{ii} - \beta c \rho) \\
\vec{J}_i' &= \vec{J}_i
\end{align*}
\]

*Plug in for \( \rho, u_{ii}, \vec{J}_i, \) and the delta functions...

\[
\begin{align*}
\rho' &= \gamma \left\{ \frac{c q \delta(\vec{x} - \vec{x}_i(t))}{c} - \frac{\vec{v} \cdot q \vec{u}_i \delta(\vec{x} - \vec{x}_i(t))}{c} \right\} \\
u_{ii}' &= \gamma \left\{ q u_{ii} \delta(\vec{x} - \vec{x}_i(t)) - \frac{\vec{v} \cdot c q \delta(\vec{x} - \vec{x}_i(t))}{c} \right\} \\
\vec{J}_i' &= q \vec{u}_i \delta(\vec{x} - \vec{x}_i(t))
\end{align*}
\]

continued on back
9.14 An alternate approach to 9.14 based on cylindrical coordinates is possible and will be provided later
\[ p' = \gamma q \delta(x-x'_0) \left\{ 1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right\} = \frac{\gamma q \delta(x-x'_0)}{\gamma (1 - \frac{\vec{v} \cdot \vec{u}}{c^2}) (1 - \frac{\vec{v} \cdot \vec{u}}{c^2})} \]

\[ = q \delta(x-x'_0) \]

\[ \rightarrow \quad u_{11}' = \gamma q \delta(x-x'_0) \left\{ u_{11} - \vec{v} \right\} = \frac{\gamma q \delta(x-x'_0)}{(1 - \frac{\vec{v} \cdot \vec{u}}{c^2}) (1 - \frac{\vec{v} \cdot \vec{u}}{c^2})} (u_{11} - \vec{v}) \]

\[ = q u_{11}' \delta(x-x'_0) \]

\[ \rightarrow \quad j'_1 = q u_{11}' \delta(x-x'_0) = q u_{11}' \left[ \frac{\delta(x-x'_0)}{\gamma (1 - \frac{\vec{v} \cdot \vec{u}}{c^2})} \right] \]

\[ = q u_{11}' \delta(x-x'_0) \]

Therefore...

We've verified that \((p, j)\) transforms as a 4-vector by showing the Lorentz transformation to a primed frame yields exactly the same form of \(p\) and \(j\), just primed:

\[ p' = q \delta(x-x'_0) \]

\[ j'_1 = q u_{11}' \delta(x-x'_0) \]
JACKSON #9.14: An antenna consists of a circular loop of wire of radius \( a \) located in the \( x-y \) plane with its center at the origin. The current in the wire is \( I = I_0 \cos \omega t = \text{Re} \{ I_0 e^{-jut} \} \).

a) Find the expressions for \( \vec{E}, \vec{H} \) in the radiation zone without approximations as to the magnitude of \( ka \). Determine the power radiated per unit solid angle.

Following Jackson Section 9.7, multipole solutions for \( \vec{E} \) and \( \vec{H} \) are found, given by Eq. 9.122:

\[
\vec{H} = \sum_{\ell,m} \left[ a_\ell(l, m) \hat{g}_{\ell} (kr) \vec{X}_{\ell m} - i a_m(l, m) \nabla \times g_{\ell} (kr) \vec{X}_{\ell m} \right] / k
\]

\[
\vec{E} = i \frac{Z_0}{k} \nabla \times \vec{H} = \frac{Z_0}{i k} \sum_{\ell,m} \left[ i a_\ell(l, m) \nabla \times \hat{g}_{\ell} (kr) \vec{X}_{\ell m} + a_m(l, m) \hat{g}_{\ell} (kr) \vec{X}_{\ell m} \right]
\]

where using Jackson convention, the following are defined...

\( f_{\ell} (kr) = A^{(\ell)} \hat{h}_{\ell}^{(1)} (kr) + A^{(\ell)} \hat{h}_{\ell}^{(2)} (kr) \)

\( g_{\ell} (kr) = B^{(\ell)} \hat{h}_{\ell}^{(1)} (kr) + B^{(\ell)} \hat{h}_{\ell}^{(2)} (kr) \)

\( \hat{h}_{\ell}^{(1,2)} \) are Hankel Functions

\( \vec{X}_{\ell m} = \frac{1}{\sqrt{\ell (\ell + 1)}} Y_{\ell m}(\theta, \phi) \)

\( \vec{l} = i (\vec{r} \times \nabla) \)

\( \vec{r} = r \hat{r} \)

\( Z_0 \) is the impedance of free space, \( \sqrt{\mu / \varepsilon} \)

\( a_\ell(l, m) \) and \( a_m(l, m) \) specify the amounts of electric and magnetic multipole fields.

**** THESE WILL BE DEFINED LATER****

AT THIS POINT: We wish to find \( \vec{E} \) and \( \vec{H} \) in the radiation zone, a.k.a. near field limit, away from the source. In this limit, \( kr \gg 1 \), and we can make some approximations to find the appropriate \( \vec{E}, \vec{H} \).
* For fields far from the source, \( \kappa r \gg 1 \); for outgoing waves at large distances, we must have no \( h^{(2)} \) ... therefore \( A_e^{(2)} = B_e^{(2)} \) 
(appropriate for a localized source like over)

**AND:** for \( \kappa r \gg 1 \), \( h^{(1)}(\kappa r) \rightarrow \frac{-i}{\kappa r} \frac{\varepsilon_{ikr}}{\kappa r} \)

**SO NOW:** In this limit, we now have parts of \( \vec{E} \) that look like...

\[
\begin{align*}
\nabla \times \{ (-i)^{l+1} \frac{\varepsilon_{ikr}}{\kappa r} \vec{X}_{lm} \} & \quad \text{**We don't want to keep anything higher than } \frac{1}{\kappa r} \text{ (} \kappa r \text{ already small)} \nabla \times \{ \varepsilon_{ikr} \vec{X}_{lm} + \varepsilon_{ikr} \nabla \times \vec{X}_{lm} \} & \quad \text{**USE: } \nabla \times (\vec{A}) = \nabla \times \vec{A} + \vec{j} \times \vec{A} \text{ to expand} \\
& = (-i)^{l+1} \left[ \varepsilon_{ikr} \left( \frac{\nabla \times \vec{X}_{lm} + \varepsilon_{ikr} \nabla \times \vec{X}_{lm}}{\kappa r} \right) \right] \\
& = (-i)^{l+1} e^{ikr} \left[ \frac{\varepsilon_{ikr}}{\kappa r} \vec{X}_{lm} + \frac{\varepsilon_{ikr} \nabla \times \vec{X}_{lm}}{\kappa r} \right] \\
& \quad \text{**DROP THIS TERM} \\
& = (-i)^{l+1} e^{ikr} \left[ \frac{\varepsilon_{ikr}}{\kappa r} \vec{X}_{lm} + \frac{\varepsilon_{ikr} \nabla \times \vec{X}_{lm}}{\kappa r} \right] \\
& \quad \text{**VECTOR-IDENTITY} \\
& \quad \Rightarrow \frac{1}{\kappa r} \left( \frac{1}{\kappa r} \right) \quad \text{**IF } \kappa r \gg 1 \Rightarrow \text{NEGLIGENCE}
\end{align*}
\]

**WE CAN NOW WRITE \( \vec{H} \) and \( \vec{E} \) EXPRESSIONS in the RADIATION ZONE**...

\[
\vec{H} = \frac{e^{ikr}}{\kappa r} \sum_{l,m} (-i)^{l+1} \left[ a_{e(l,m)} \vec{X}_{lm} + a_{m(l,m)} \hat{r} \times \vec{X}_{lm} \right]
\]

\[
\vec{E} = \frac{e^{ikr}}{\kappa r} \sum_{l,m} (-i)^{l+1} \left[ a_{e(l,m)} \hat{r} \times \vec{X}_{lm} + a_{m(l,m)} \vec{X}_{lm} \right]
\]

**But what are \( a_{e(l,m)} \) and \( a_{m(l,m)} \)? Following the Jackson Section 9.10 derivation, we come to the electric and magnetic multipole expressions as expressed in terms of potential charge, current, and intrinsic magnetization sources. They are given by Eq. 9.167 and Eq. 9.168...**

Continued on next page
\[ a_e(l, m) = \frac{k^2}{i \sqrt{l(l+1)}} \int \frac{\partial}{\partial r} \left[ r j(e(kr)) \right] + i k (\mathbf{F} \cdot \mathbf{j}) j(e(kr)) - i k \mathbf{V} \cdot (\mathbf{F} \times \mathbf{M}) \ j(e(kr)) \ d^3 x \]

\[ a_m(l, m) = \frac{k^2}{i \sqrt{l(l+1)}} \int \frac{1}{\partial r} \left[ (\mathbf{F} \cdot \mathbf{j}) (\mathbf{j}(e(kr)) - k^2 (\mathbf{F} \cdot \mathbf{M}) \ j(e(kr)) \right] \ d^3 x \]

* However: In our problem, we're dealing with a loop current source, there is no charge or intrinsic magnetization sources.

So... \( \rho = 0, \ \mathbf{M} = 0, \ \mathbf{J} \neq 0. \)

* This reduces our coefficients accordingly...

\[ a_e(l, m) = \frac{k^2}{l(l+1)} \int \mathbf{V} \cdot (\mathbf{F} \cdot \mathbf{j}) j(e(kr)) \ d^3 x \]

\[ a_m(l, m) = \frac{k^2}{i \sqrt{l(l+1)}} \int \mathbf{V} \cdot (\mathbf{F} \cdot \mathbf{j}) j(e(kr)) \ d^3 x \]

* SO NOW: We have expressions for \( \mathbf{H}_i \) and \( \mathbf{E}_i \), as well as \( a_e \), \( a_m \). But, we need to specify them further, for the given problem... what is \( \mathbf{J} \)?

* Recall our current loop source...

  * Our current source, \( I(t) = I_0 \cos \omega t = \text{Re} \{ I_0 e^{-i\omega t} \} \)
    points directionally in the \( \phi \) direction, and is located at \( r = a, \ \theta = \pi/2 \) (or \( \cos \theta = 0 \))

* Since \( \mathbf{J} \) is defined via \( \mathbf{I} = \int \mathbf{J} \ d\alpha \), it's natural to take \( \mathbf{J} = \phi \delta(r-a) \delta(\cos \theta) \), up to a constant. Dropping time dependencies to consider amplitudes...

\[ I_0 = \int N \delta(r-a) \delta(\cos \theta) \ d\alpha \]

\[ = Na \quad \text{therefore:} \quad \mathbf{J}(r) = \frac{I_0}{a} \delta(r-a) \delta(\cos \theta) \]

continued on back
SO NOW: Work on getting expressions for $\alpha_{E(l,m)}$ and $\alpha_{\mu(l,m)}$...

$\mathbf{F} \cdot \mathbf{J} = \nabla \cdot (r \mathbf{J} (r) \mathbf{p}^2) = \text{ZERO} \quad \Rightarrow \quad \text{this implies that } \alpha_{E(l,m)} \text{ is ZERO}

$\nabla \cdot (\mathbf{r} \times \mathbf{J}) = \nabla \cdot (r \mathbf{J} (r) \mathbf{p}^2) = \nabla \cdot (r \mathbf{J} (r) \mathbf{p}^2) = \frac{-1}{r \sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \mathbf{J} (r) \right] j_e(r) r^2 dr d(cos \theta) d\phi

\Rightarrow \text{this implies that } \alpha_{\mu(l,m)} \text{ is NONZERO}

\[
\alpha_{E(l,m)} = 0
\]

\[
\alpha_{\mu(l,m)} = \frac{k^2}{i \sqrt{e(l+1)}} \sum \frac{Y_m^*}{\delta (cos \theta)} \left[ \sin \theta \mathbf{J} (r) \right] j_e(r) r^2 dr d(cos \theta) d\phi
\]

But: $\mathbf{J} (r, \theta, t) = \frac{I(t)}{\delta (r-a)} \delta (cos \theta)

\[
= \frac{I(t)}{\delta (r-a)} \frac{k^2}{\sqrt{e(l+1)}} \sum \frac{Y_m^*}{\delta (cos \theta)} \frac{\partial}{\partial \theta} \left[ \sin \theta \delta (cos \theta) \right] j_e(r) r^2 dr d(cos \theta) d\phi
\]

**NOTE:** Our system is the same for all $\phi$; it is AZIMUTHALLY SYMMETRIC

$m = 0, \quad Y_m \Rightarrow Y_0 = \sqrt{\frac{2l+1}{4\pi}} P_l(cos \theta)$

\[
= \delta_{m,0}(2\pi) I(t) k^2 \frac{a}{\sqrt{e(l+1)}} \sqrt{\frac{2l+1}{4\pi}} \sum P_l(cos \theta) \frac{\partial}{\partial \theta} \left[ \sin \theta \delta (cos \theta) \right] d(cos \theta)
\]

$\Rightarrow \text{this will integrate to ZERO}

Via the definition of the derivative of the delta function

$\Rightarrow \text{continued on next page}$
But what are \( \mathbf{\hat{r} \times \hat{x}_0} \) and \( \hat{x}_0 \)?

\[
\hat{x}_0 = \frac{1}{\sqrt{\ell (\ell + 1)}} \mathbf{Y}_\ell 0 (\theta) = \frac{1}{i \sqrt{\ell (\ell + 1)}} (\mathbf{\hat{r} \times \hat{\theta}}) Y_\ell 0 (\theta) = \frac{1}{i \sqrt{\ell (\ell + 1)}} \sqrt{2\ell + 1} (\mathbf{\hat{r} \times \hat{\theta}}) P_\ell (\cos \theta)
\]

\[
= \frac{1}{i \sqrt{\ell (\ell + 1)}} \sqrt{2\ell + 1} \left\{ \mathbf{\hat{r} \times \hat{\theta}} \right\} P_\ell (\cos \theta)
\]

\[
= \frac{1}{i \sqrt{\ell (\ell + 1)}} \sqrt{2\ell + 1} \left\{ \frac{\partial}{\partial \theta} \hat{\phi} + \frac{\partial}{\partial \phi} \hat{\theta} \right\} P_\ell (\cos \theta)
\]

\[
\mathbf{\hat{r} \times \hat{x}_0} = \frac{1}{i \sqrt{\ell (\ell + 1)}} \sqrt{2\ell + 1} \left\{ \frac{\partial}{\partial \theta} \hat{\phi} - \frac{\partial}{\partial \phi} \hat{\theta} \right\} P_\ell (\cos \theta)
\]

Using the above we now have fully fleshed out expressions for \( \mathbf{\hat{H}} \) and \( \mathbf{\hat{E}} \).

\[
\mathbf{\hat{H}} = -e^{ikr} \sum_{l = 0} \left\{ \mathbf{\hat{r} \times \hat{\theta}} \right\} P_\ell (\cos \theta) \frac{\partial}{\partial \theta} \hat{\phi} + \frac{\partial}{\partial \phi} \hat{\theta}
\]

\[
\mathbf{\hat{E}} = \sum_{l = 0} \left\{ \mathbf{\hat{r} \times \hat{\theta}} \right\} P_\ell (\cos \theta) \frac{\partial}{\partial \theta} \hat{\phi} - \frac{\partial}{\partial \phi} \hat{\theta}
\]

SO NOW: To find the power radiated per unit solid angle.....

We know via Jackson, Eq. 9.21, that the power radiated per unit solid angle time averaged is...

\[
\frac{dp}{d\Omega} = \frac{1}{2} \text{Re} \left[ \mathbf{r^2} \cdot \mathbf{E} \times \mathbf{H^*} \right]
\]

Define the following for simplicity...

\[
\mathbf{\hat{H}} = -\mathbf{\hat{H}_0} \hat{\theta}
\]

\[
\mathbf{\hat{E}} = \mathbf{\hat{E}_0} \hat{\phi}
\]

where

\[
\mathbf{\hat{H}_0} = e^{ikr} \sum_{l = 0} \left\{ \mathbf{\hat{r} \times \hat{\theta}} \right\} P_\ell (\cos \theta) \frac{\partial}{\partial \theta} \hat{\phi} - \frac{\partial}{\partial \phi} \hat{\theta}
\]

\[
\mathbf{\hat{E}_0} = \sum_{l = 0} \left\{ \mathbf{\hat{r} \times \hat{\theta}} \right\} P_\ell (\cos \theta) \frac{\partial}{\partial \theta} \hat{\phi} + \frac{\partial}{\partial \phi} \hat{\theta}
\]

continued on next page
\[ a_m(l, n) = \epsilon_m \sqrt{\frac{2l+1}{4\pi}} \frac{\gamma_l(l+1)}{\sqrt{l(l+1)}} \sqrt{\frac{2l+1}{4\pi}} \left\{ \begin{array}{l} \lambda P_l(\cos \theta) \sin \theta \pmod{0} \\
\cos \theta = 0 \\
\sin \theta = 1 \end{array} \right. \]

THIS TELLS US...

- \[ q_e(l, m) = 0 \]
- \[ a_m(l, m) = \xi_m(l) \frac{\gamma_l(l+1)}{\sqrt{l(l+1)}} \sqrt{\frac{2l+1}{4\pi}} \left\{ \begin{array}{l} \lambda P_l(\cos \theta) \sin \theta \pmod{0} \\
\cos \theta = 0 \\
\sin \theta = 1 \end{array} \right. \]

**NOTE:** \( P_l(\cos \theta) \) are polynomials that are odd or even in powers of \( \cos \theta \), taking a derivative will switch those powers, meaning that \( P_l(0) \) for odd \( l \) is \( 0 \).

but \( \lambda P_l(0) \) will be \( 0 \) for even \( l \).

**SO IN FACT:**

\[ a_m(l, m) = \left\{ \begin{array}{l} 0 \text{ for } l \text{ even} \\
\xi_m(l) \frac{\gamma_l(l+1)}{\sqrt{l(l+1)}} \sqrt{\frac{2l+1}{4\pi}} \left\{ \begin{array}{l} \lambda P_l(\cos \theta) \sin \theta \pmod{0} \\
\cos \theta = 0 \\
\sin \theta = 1 \end{array} \right. \text{ for } l \text{ odd} \end{array} \right. \]

**NOW PLUGGING THESE INTO OUR EXPRESSIONS for \( H \):**

\[ \dot{H} = \frac{e^{i k r}}{k r} \sum_{l, m} (-i)^{l+1} \left\{ \begin{array}{l} 0 \text{ even } l \\
\xi_m(l) \frac{\gamma_l(l+1)}{\sqrt{l(l+1)}} \sqrt{\frac{2l+1}{4\pi}} \left\{ \begin{array}{l} \lambda P_l(\cos \theta) \sin \theta \pmod{0} \\
\cos \theta = 0 \\
\sin \theta = 1 \end{array} \right. \text{ for } l \text{ odd} \end{array} \right. \left( \hat{r} \times \hat{x}_m \right) \]

\[ \dot{E} = \frac{Z_0 e^{i k r}}{k r} \sum_{l, m} (-i)^{l+1} \left\{ \begin{array}{l} 0 \text{ even } l \\
\xi_m(l) \frac{\gamma_l(l+1)}{\sqrt{l(l+1)}} \sqrt{\frac{2l+1}{4\pi}} \left\{ \begin{array}{l} \lambda P_l(\cos \theta) \sin \theta \pmod{0} \\
\cos \theta = 0 \\
\sin \theta = 1 \end{array} \right. \text{ for } l \text{ odd} \end{array} \right. \hat{x}_m \]

\[ \ddot{H} = \frac{e^{i k r}}{k r} \sum_{l, m} (-i)^{l+1} \frac{i \pi a_i(l) k^2 \gamma_m(l)}{\sqrt{l(l+1)}} \sqrt{\frac{2l+1}{4\pi}} \left\{ \begin{array}{l} \lambda P_l(\cos \theta) \sin \theta \pmod{0} \\
\cos \theta = 0 \\
\sin \theta = 1 \end{array} \right. \left( \hat{r} \times \hat{x}_m \right) \]

\[ \ddot{E} = \frac{Z_0 e^{i k r}}{k r} \sum_{l, m} (-i)^{l+1} \frac{i \pi a_i(l) k^2 \gamma_m(l)}{\sqrt{l(l+1)}} \sqrt{\frac{2l+1}{4\pi}} \left\{ \begin{array}{l} \lambda P_l(\cos \theta) \sin \theta \pmod{0} \\
\cos \theta = 0 \\
\sin \theta = 1 \end{array} \right. \hat{x}_m \]

Continued on back
\[
\begin{align*}
\text{THEN: } \frac{dP}{d\Omega} &= \frac{1}{2} \text{Re} \left[ r^2 \hat{r} \cdot \left( Z_0 H_0 \hat{\phi} \right) \times \left( -H_0^* \hat{\theta} \right) \right] \\
&= \frac{1}{2} \text{Re} \left[ r^2 \hat{r} \cdot \left( Z_0 |H_0|^2 \hat{\phi} \times \theta \hat{\theta} \right) \right] \\
&= \frac{1}{2} r^2 Z_0 |H_0|^2
\end{align*}
\]

LEAVING US WITH...

\[
\text{POWER RADIATED per UNIT SOLID ANGLE}
\]
\[
\frac{dP}{d\Omega} = \frac{1}{2} r^2 Z_0 |H_0|^2
\]
\[
= \frac{Z_0}{2k^2} \sum_{l=0}^{\infty} \frac{(ci)^{l+1}}{2l+1} \frac{e^{i(l+1)kz}}{l(l+1)} \left( \frac{\partial P_e(\cos \theta)}{\partial \cos \theta} \right)^2 \left( \frac{\partial P_m(\cos \theta)}{\partial \cos \theta} \right)^2
\]

b) What is the lowest nonvanishing multipole moment \((Q_{1,m} \text{ or } M_{1,m})\)?
Evaluate this moment in the limit \(k a \ll 1\).

\[\text{considering we've already established for our problem that the } Q_{L,M} \text{'s are ZERO, there's no way the associated electric multipole moments, } Q_{e,m} \text{'s, could be the lowest NONVANISHING}\]

\[\text{As for the magnetic multipole moments, } M_{L,m} \text{'s, we've established } a_{L,m} \text{ is ZERO for EVEN } l\]

\[\text{THEREFORE } \]
\[\Rightarrow M_{11} \text{ is the lowest nonvanishing multipole moment.}\]

\[\text{SO NOW: We wish to evaluate } M_{11}. \text{ If we follow Jackson Section 9.10 once again, they approximate } a_{L} \text{ and } a_{m} \text{ in the } k a \ll 1 \text{ limit to establish definitions of } M_{L,m} \text{ and } Q_{e,m}\]
Since we're concerned with $M_{10}$, go to the Am focused point...

Jackson Eq. 9.141 and Eq. 9.172 tell us...

$$\begin{align*}
L \cdot a_m(l,m) & \sim \frac{i k^{l+2}}{(2l+1)!!} \sqrt{\frac{l+1}{l}} M_{lm} = \frac{i k^{l+2}}{(2l+1)!!} \sqrt{\frac{l+1}{l}} \left\{ -\frac{1}{l+1} \int r^2 Y_{lm}^* \nabla \cdot (r \times \hat{j}) d^3 r \right\} \\
\text{But: in part (a) we found } a_m(l,m) & = \sum_{n=0}^{\infty} \frac{i^m 2^{m-1} \Gamma(n+1) k^n j(n) \gamma^{2l+1}}{\sqrt{\pi} (l+1)!!} \left\{ 2 \left( \frac{\alpha}{\beta} \right) \right\} \\
& \quad \text{for odd } l, \quad \forall \gamma \geq 0, \quad \text{we also want to let } \\
& \quad \frac{ka}{\gamma} << 1, \quad \text{where in this limit, } j_0(ka) \to \frac{ka}{\gamma} + \ldots \\
\end{align*}$$

\[ \implies \text{T A K E } l = 1, m = 0; \quad P_l(\cos \Theta) = \cos \Theta; \quad \text{we also want to let } \]
\[ \frac{ka}{\gamma} << 1, \quad \text{where in this limit, } j_0(ka) \to \frac{ka}{\gamma} + \ldots \]

$$\begin{align*}
L \cdot a_{10}(1,0) & \sim \frac{i \gamma^{2+2}}{3!!} \frac{ka}{\gamma} \right\{ -\frac{1}{l+1} \int r^2 Y_{10}^* \nabla \cdot (r \times \hat{j}) d^3 r \right\} \\
& = \frac{i k^2}{3!!} \left( \frac{\alpha}{\beta} \right) M_{10} \quad \text{(according to the above...)}
\end{align*}$$

\[ \text{Clearly then...} \]

\[ M_{10} = \sqrt{\frac{2}{4\pi}} \frac{I_0 \pi a^2}{4\pi} \]
JACKSON #11.3: Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity \( V = \frac{V_1 + V_2}{1 + \frac{V_1 V_2}{C^2}} \).

We can define the unprimed, primed, and double primed frames as follows:

- \( x_0, x_1, x_2, x_3 \) is REST FRAME \( \hat{x} \)
- \( x'_0, x'_1, x'_2, x'_3 \) is moving at velocity \( V_1 \) relative to rest frame \( \hat{x} \)
- \( x''_0, x''_1, x''_2, x''_3 \) is moving at velocity \( V_2 \) relative to primed frame \( \hat{x}' \)

*These are related via Lorentz transformation, where by transforming in the same direction the \( x-x' \) relations are identical in form to the \( x-x'' \) relations. THEY ARE:

\[
\begin{align*}
\text{UNPRIME-PRIME} & \quad \text{PRIME-DOUBLE PRIME} \\
X'_0 &= \gamma_1 (X_0 - \beta_1 X_1) & X''_0 &= \gamma_2 (X'_0 - \beta_2 X'_1) \\
X'_1 &= \gamma_1 (X_1 - \beta_1 X_0) & X''_1 &= \gamma_2 (X'_1 - \beta_2 X'_0) \\
X'_2 &= X_2 & X''_2 &= X'_2 \\
X'_3 &= X_3 & X''_3 &= X'_3
\end{align*}
\]

where \( \gamma_1 = \frac{1}{\sqrt{1 - \beta_1^2 c^2}} \) where \( \gamma_2 = \frac{1}{\sqrt{1 - \beta_2^2 c^2}} \)

\( \beta_1 = \frac{V_1}{c} \)

\( \beta_2 = \frac{V_2}{c} \)

**So now:** use these relations to create a transformation from the unprimed \( \rightarrow \) double primed frame, in an attempt to make this as one transformation.

\[
\begin{align*}
X''_0 &= \gamma_2 \left( \gamma_1 (X_0 - \beta_1 X_1) - \beta_2 \gamma_1 (X_1 - \beta_1 X_0) \right) \\
X''_1 &= \gamma_2 \left( \gamma_1 (X_1 - \beta_1 X_0) - \beta_2 \gamma_1 (X_0 - \beta_1 X_1) \right) \\
X''_2 &= X_2 \\
X''_3 &= X_3
\end{align*}
\]

*REARRANGE NOW to group up \( X_0, X_1 \) terms *

\[\text{continued on back}\]
UNPRIME - DOUBLE PRIME

\[ x_0'' = (\gamma_1 \gamma_2 + \gamma_1 \gamma_2 \beta_1 \beta_2) x_0 - (\gamma_1 \gamma_2 \beta_1 + \gamma_1 \gamma_2 \beta_2) x_1 \]
\[ x_1'' = (\gamma_1 \gamma_2 + \gamma_1 \gamma_2 \beta_1 \beta_2) x_1 - (\gamma_1 \gamma_2 \beta_1 + \gamma_1 \gamma_2 \beta_2) x_0 \]
\[ x_2'' = x_2 \]
\[ \gamma_2 = \beta \]

\[ X_2'' = X_2 \]

where \( \gamma_i = \frac{1}{\sqrt{1 - v_i^2/c^2}} \) and \( \beta_i = \frac{v_i}{c} \)

**BUT NOW NOTE:** This looks like a single Lorentz transformation to a double-primed frame moving at speed \( \mathbf{v} \) relative to the unprimed frame defined via a new \( \gamma \) and \( \beta \)

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v_2}{c^2}}} \]
\[ \beta = \frac{v_2/c}{\sqrt{1 - \frac{v_2}{c^2}}} \]

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v_2}{c^2}}} \]

\[ \gamma \gamma_2 (1 + \beta_1 \beta_2) \]

Now use this to solve for this speed, \( \mathbf{v} \)

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v_2}{c^2}}} \]

\[ \gamma \gamma_2 (1 + \beta_1 \beta_2) = (\frac{1}{\sqrt{1 - \frac{v_2}{c^2}}} \frac{1}{\sqrt{1 - \frac{v_2}{c^2}}}) (1 + \frac{v_1 \gamma_2}{c^2}) \]

Square both sides, solve for \( \gamma \)

\[ \frac{1}{1 - \frac{v_2^2}{c^2}} = \frac{(1 + \frac{v_1 \gamma_2}{c^2})^2}{(1 - \frac{v_2}{c^2})(1 - \frac{v_2^2}{c^2})} \]

\[ v = \sqrt{\frac{c^2 \left( 1 - \frac{v_2^2}{c^2} \right)^2}{1 - \frac{v_2}{c^2} \left( 1 - \frac{v_2^2}{c^2} \right)}} \]

\[ v = \sqrt{\frac{(v_1 + v_2)^2}{(1 + v_1 v_2/c^2)^2}} = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \]

WE'VE SHOWN THAT: Two Lorentz transformations in the same direction is equivalent to a single one, with the velocity...

\[ v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \]
JACKSON #11.19: A particle of mass \( M \) and 4-momentum \( P \) decays into two particles of mass \( m_1 \) and \( m_2 \).

a) Use the conservation of energy and momentum in the form, \( p_2 = P - p_1 \), and the invariance of scalar products of 4-vectors to show that the total energy of the first particle in the rest frame of the decaying particle is...

\[
E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}
\]

and that \( E_2 \) is obtained by interchanging \( m_1 \) and \( m_2 \).

**TAKING \( C = 1 \) THROUGHOUT**

The 4-momentum of some particle with energy \( E \) and 3D momentum \( \mathbf{p} \) is \((E, \mathbf{p}) = (E, p_x, p_y, p_z)\).

**IN OUR CASE**: In the rest frame of the decaying particle, before decaying the particle will have no momentum and only its rest energy. Post decay, the particles will each have total energies \( E_1 \) and \( E_2 \), and momenta which are equal in magnitude but opposite in direction (conservation of 3D momentum).

**SO ⇒ Before**: \( P = (M, \mathbf{0}) \); After: \( p_1 = (E_1, \mathbf{p}) \) and \( p_2 = (E_2, -\mathbf{p}) \).

*Conservation of Energy and momentum in terms of 4-momenta will give:

\[
P = p_1 + p_2 \quad \text{REARRANGED, we then have...}
\]

→ \( p_1 = P - p_2 \) *WE NOW WANT TO TAKE the SCALAR PRODUCT of the 4-VECTORS on EACH SIDE WITH THEMSELVES, i.e. SQUARE BOTH SIDES

→ \( p_2 = P - p_1 \)
The scalar product of two 4-vectors \( \mathbf{A} = (A_0, \vec{A}) \) and \( \mathbf{B} = (B_0, \vec{B}) \) is:

\[
\mathbf{B} \cdot \mathbf{A} = B_0 A_0 - \vec{B} \cdot \vec{A}
\]

Using their to square both sides of \( \mathbf{p}_1 = \mathbf{P} - \mathbf{p}_2 \) and \( \mathbf{p}_2 = \mathbf{P} - \mathbf{p}_1 \)...

\[
\begin{align*}
\mathbf{p}_1 \cdot \mathbf{p}_1 &= E_1^2 - p^2 = P \cdot P + p_2 \cdot p_2 - 2P \cdot p_2 \\
&= M^2 + E_2^2 - p^2 - 2ME_2 \\
\mathbf{p}_2 \cdot \mathbf{p}_2 &= E_2^2 - p^2 = P \cdot P + p_1 \cdot p_1 - 2P \cdot p_1 \\
&= M^2 + E_1^2 - p^2 - 2ME_1
\end{align*}
\]

These yield:

\[
\begin{align*}
E_1^2 &= M^2 + E_2^2 - 2ME_2 \\
E_2^2 &= M^2 + E_1^2 - 2ME_1
\end{align*}
\]

Solve for \( E_2 \) in (1) and \( E_1 \) in (2) to yield...

\[
\begin{align*}
E_1 &= \frac{M^2 + E_1^2 - E_2^2}{2M} \\
E_2 &= \frac{M^2 + E_2^2 - E_1^2}{2M}
\end{align*}
\]

\[\text{BUT: We know } E_1^2 = p_i^2 + m_i^2 \]

Letting \( c = 1 \); in this case \( p_1 = p_2 = p \)

So \( E_1^2 = p^2 + m_1^2 \)

\[
E_2^2 = p^2 + m_2^2
\]

Subtract these and plug in to expr on the left.

Therefore...

\[
\begin{align*}
E_1 &= \frac{M^2 + m_i^2 - m_j^2}{2M} \quad \text{and} \quad E_2 = \frac{M^2 + m_j^2 - m_i^2}{2M}
\end{align*}
\]

b) Show that the kinetic energy, \( T_i \) of the \( i \)th particle in the same frame is...

\[
T_i = \Delta m \left( \frac{1 - m_i - \Delta m}{M} \right) \quad \text{where} \quad \Delta m = M - m_i - m_j
\]

We know the total energy for either particle, \( E_i \) or \( E_j \), is composed of the kinetic energy of the particle and the particle's rest energy. So, with \( c = 1 \), and in general for either particle...

\[E_i = T_i + m_i\]
\[ T_i = E_i - m_i \quad \text{where} \quad E_i = \frac{M^2 + m_i^2 - m_j^2}{2M}, \quad \text{and} \quad \text{index} \ j \ \text{is the} \ \text{"other" \ particle} \]

\[ T_i = E_i - m_i = \frac{M^2 + m_i^2 - m_j^2}{2M} - m_i \]

\[ = \frac{M^2 + m_i^2 - m_j^2 - 2MM_i}{2M} \]

*Notice:* \( (M-m_i+m_j)(M-m_i-m_j) = M^2 + m_i^2 - m_j^2 - 2MM_i \)

\[ = \frac{(M-m_i-m_j)(M+m_j-M-m_i+m_i+m_j)}{2M} \]

\[ = \frac{(M-m_i-m_j)(2M-2m_i-[M-m_i-m_j])}{2M} \]

\[ \text{Note that} \ M-m_i-m_j \ \text{is the same regardless if} \ i, j \ \text{is 1,2 or 2,1. So: this can be written as} \]

\[ \Delta M = M - m_i - m_j \]

\[ \text{THEREFORE:} \ \text{we can clearly then write...} \]

\[ T_i = \Delta M \left\{ 1 - \frac{m_i - \Delta M}{M} \right\} \]

\[ \text{where} \ \Delta M = M - m_i - m_j \]

**c)** The charged pion meson \( (M = 139.6 \text{ MeV}) \) decays into a mu-meson \( (m_i = 105.7 \text{ MeV}) \) and a neutrino \( (m_j = 0) \). Calculate the kinetic energies of the mu-meson and neutrino in the pion meson's rest frame.

**Note:** The kinetic energies we want can be found using the expression for \( T_i \), from part (a), found in the rest frame of the decaying particle.

**HERE:** \( M = 139.6 \text{ MeV} \); \( m_i = 105.7 \text{ MeV} \); \( m_j = 0 \)

**(the pion meson)** \( \quad \text{(the mu-meson)} \quad \text{(the neutrino)} \)

where we seek \( T_1 \) and \( T_2 \).
\[ \Delta M = \Delta M - m_1 - m_2 \ldots \]
\[ = 33.9 \text{ MeV} \]

**NOW:** Evaluate \( T_1 \) and \( T_2 \)...

\[ T_1 = \Delta M \left( 1 - \frac{m_1 - \Delta M}{M} \right) = \left( \frac{33.9 \text{ MeV}}{2(139.6)} \right) \left( 1 - \frac{105.7 - 33.9}{139.6} \right) \]
\[ = 4.11 \text{ MeV} \]

\[ T_2 = \Delta M \left( 1 - \frac{m_2 - \Delta M}{M} \right) = \left( \frac{33.9 \text{ MeV}}{2(139.6)} \right) \left( 1 - 0 \right) \]
\[ = 29.78 \text{ MeV} \]

For the \( \gamma \)-meson: 4.11 MeV; For the \( \nu \)-neutrino: 29.78 MeV