Lecture 21

April 21, 2020
Periodic Structures

Filters
Gratings
Slow Wave Structures
  particle accelerators
  Cherenkov microwave generators
Metamaterials

Floquet Theory
Floquet Theory

\[ E(z,t) = \text{Re}\left\{ \hat{E}(z)e^{-i\omega t} \right\} \]

\[ \frac{\partial^2}{\partial z^2} \hat{E}(z) + \frac{\omega^2}{c^2} \varepsilon_{\text{rel}}(z) \hat{E}(z) = 0 \]

\[ \varepsilon_{\text{rel}}(z) = \varepsilon_{\text{rel}}(z + L) \]

Special case - homogeneous
\[ \frac{\partial \varepsilon_{\text{rel}}}{\partial z} = 0 \]

\[ \hat{E}(z) = \hat{E}_0 \exp(ikz) \]

\[ \omega(k) = \pm kc / \sqrt{\varepsilon_{\text{rel}}} \]

Time harmonic, spatially dependent field

Inhomogeneous relative dielectric

Dielectric is spatially periodic

[Diagram showing dispersion relation with \( \omega \) vs. \( k \) in the plane, slope = \( c/\sqrt{\varepsilon_{\text{rel}}} \).]
Spatially Varying Case

\[ E(z,t) = \text{Re}\{\hat{E}(z)e^{-i\omega t}\} \]

\[
\frac{\partial^2}{\partial z^2} \hat{E}(z) + \frac{\omega^2}{c^2} \varepsilon_{\text{rel}}(z) \hat{E}(z) = 0
\]

\[ \varepsilon_{\text{rel}}(z) = \varepsilon_{\text{rel}}(z + L) \]

\[ \hat{E}(z) = \hat{E}_0(k,z) \exp(ikz) \]

\[ \hat{E}_0(k,z) = \hat{E}_0(k,z + L) \]

\[ \omega(k) = \omega(k + k_0) \]

\[ k_0 = \frac{2\pi}{L} \]
Smith Island Cake

\[
\begin{pmatrix}
E(d) \\
H(d)
\end{pmatrix}
=egin{pmatrix}
\cos \theta & iZ \sin \theta \\
iZ^{-1} \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
E(0) \\
H(0)
\end{pmatrix}
\]

\[Z = \sqrt{\mu_0/\varepsilon}\]

\[\theta = kd = \omega \sqrt{\varepsilon \mu_0 d}\]

\[
\begin{pmatrix}
E(z+d_1+d_2) \\
H(z+d_1+d_2)
\end{pmatrix}
= \lambda
\begin{pmatrix}
E(z) \\
H(z)
\end{pmatrix}
\]

\[
\lambda^2 + b \lambda + 1 = 0
\]

\[b = (\frac{2 \varepsilon d}{\omega_0}) \sin \theta \sin \phi - 2 \omega_0 d \cos \theta \cos \phi \]

\[
\lambda = -\frac{b \pm \sqrt{b^2 - 4}}{2}
\]
\[
\lambda^* = \frac{-b + \sqrt{b^2 - 4}}{2}
\]
\[
\lambda \lambda^* = \left(-\frac{b + \sqrt{b^2 - 4}}{2}\right)\left(-\frac{b - \sqrt{b^2 - 4}}{2}\right) = \frac{b^2 - \sqrt{b^2 - 4}}{4}
\]
\[ \theta = kd = \omega \sqrt{\varepsilon \mu_0 d} \]

\[ \lambda = e^{ik(d_1 + d_2)} \]

\[ \cos\left[ k(d_1 + d_2) \right] = \cos(\theta_1 + \theta_2) - \frac{\Delta}{2} \sin(\theta_1)\sin(\theta_2) \]

\[ \Delta = \left( \frac{Z_1 - Z_2}{Z_1 Z_2} \right)^2 \]

Special case: \( \theta_1 = \theta_2 \)

\[ \cos(\theta_1 + \theta_2) = \frac{\cos\left[ k(d_1 + d_2) \right] + \Delta / 4}{1 + \Delta / 4} \]

\[ \theta_1 + \theta_2 = \frac{\omega}{c} \left( d_1 \sqrt{\varepsilon_1} + d_2 \sqrt{\varepsilon_2} \right) \]

**Solutions**

\[ \lambda^2 + b \lambda + 1 = 0 \]

\[ b = \left( \frac{d_1}{d_2}, \frac{d_2}{d_1} \right) \]

\[ \lambda = \frac{-b \pm \sqrt{b^2 - 4}}{2} \]

\[ \lambda_+ \lambda_- = \left( \frac{-b + \sqrt{b^2 - 4}}{2} \right) \left( \frac{-b - \sqrt{b^2 - 4}}{2} \right) \]

\[ = \frac{b^2 - \sqrt{b^2 - 4}}{4} = 1 \]

\[ \text{Stop Band} \]
Continuous Variations

**Mathieu Equation**

\[
\left[ \frac{d^2}{dz^2} + \frac{\omega^2}{\xi} (1 + \xi \cos kz) \right] \hat{E}(z) = 0 \quad k_0 = \frac{2\pi}{L}
\]

Write \( \hat{E}(z) = e^{ikz} \hat{E}_0(z, k) \)

\[
\hat{E}_0 = \sum_{n=-\infty}^{\infty} \hat{E}_n \exp(ik_n z) \quad k_1 = nk_0
\]

Fourier Series
Solution by Fourier Series

\[ \left[ \frac{d^2}{dz^2} + \frac{\omega^2}{c^2} \left( 1 + \frac{\cos k_0 z}{z} \right) \right] E(z) = 0 \]

\[ \int_0^L \frac{dz}{L} \exp(-i(k+nk_0)z) \cdot \{ \text{Mathieu Equation} \} = 0 \]

\[ \left[ - (k+nk_0)^2 + \frac{\omega^2}{c^2} \right] E_m + \frac{\omega^2}{c^2} \sum_{m} \int_0^L \frac{dz}{L} e^{-i k_0 z} \cos k_0 z E_m e^{i h_m z} \]

Note

\[ \int_0^L \frac{dz}{L} \cos k_0 z \exp[i k_0 z (m-n)] \]

\[ = \begin{cases} \frac{1}{2} & \text{if } m = n \pm 1 \\ 0 & \text{otherwise} \end{cases} \]
\[
\left[ \frac{\omega^2}{c^2} - (k+nk_0)^2 \right] E_n + \frac{\omega^2}{c^2} \frac{\delta}{2} \left[ E_{n+1} + E_{n-1} \right] = 0
\]

Note: if \( \omega(k) \) is a solution with \( E_n \)

Then \( k \rightarrow k + k_0 \), \( n \rightarrow n - 1 \) is also a solution \( \omega(k) = \omega(k + k_0) \)
\[
\left[ \frac{\omega^2}{c^2} - (k+n\kappa_0)^2 \right] E_n + \frac{\omega^2}{c^2} \delta \left[ E_{n+1} + E_{n-1} \right] = 0
\]

Approximate solutions for \(|\delta| < < 1\)

For each \( n \)
\( \frac{\omega^2}{c^2} = (k+n\kappa_0)^2 \) is a solution
\( E_n \neq 0 \)
\( E_{n+1} \text{ or } E_{n-1} = 0 \).

Pattern repeats with period \( \kappa_0 \).
Solution breaks down when lines cross

\[ n = 0 \]
\[ \left[ \frac{\omega^2}{c^2} - k^2 \right] E_0 + \frac{\omega^2}{c^2} \delta \frac{1}{2} E_1 = 0 \]

\[ n = -1 \]
\[ \left[ \frac{\omega^2}{c^2} - (k-k_0)^2 \right] E_{-1} + \frac{\omega^1}{c^2} \delta E_0 = 0 \]

Combine
\[ \left[ \frac{\omega^2}{c^2} - k^2 \right] \left[ \frac{\omega^1}{c^2} - (k-k_0)^2 \right] = \left( \frac{\omega^2}{c^2} \delta \right)^2 \]
Combine

\[
\left[ \frac{\omega^2}{c^2} - k^2 \right] \left[ \frac{\omega^2}{c^2} - (k-k_0)^2 \right] = \left( \frac{\omega^2 - \delta^2}{c^2} \right)^2
\]

Let \( \omega = \frac{k_0 c}{2} + sw \) \( k = \frac{k_0}{2} + sk \)

\[
\begin{bmatrix}
\frac{k_0 c}{c^2} - k_0 sk
\end{bmatrix}
\begin{bmatrix}
\frac{k_0 c}{c^2} + k_0 sk
\end{bmatrix}
= \frac{\delta^2}{2^2} (k_0 c)^2
\]

\[
\begin{bmatrix}
\frac{s^2}{c} - sk
\end{bmatrix}
\begin{bmatrix}
\frac{s^2}{c} + sk
\end{bmatrix}
= \frac{s^2}{64} k_0^2
\]
\[ \left( \frac{\delta \omega}{c} \right)^2 = 8k^2 + \frac{k_0^2}{64} \delta^2 \]

\[ \text{gap} = \frac{2c}{\delta} \frac{k_0 \delta}{8} = \frac{6}{4} k_0 c \]

**Avoided Crossing**

**Gap is like band gap for**

**electron in a crystal structure**
Grating

\[ \nabla^2 \hat{E}(x, z) + \frac{\omega^2}{c^2} \hat{E}(x, z) = 0 \]

Boundary Condition \( \hat{E}(x = d(z), z) = 0 \)

\[ d(z) = \sum_m d_m e^{imk_0 z} \]
$\hat{E} = \hat{E}_{inc} \exp\left[i \frac{k_z z - ik_x x}{\lambda}\right]$

$k_z = \frac{\omega}{c} \sin \theta_i$

$k_x = \frac{\omega}{c} \cos \theta_i$
Reflected Waves

\[ \hat{E}_{\text{ref}} = \sum_n \hat{E}_n \exp \left[ ik_2 z + i m k_0 z^2 + ik_{xn} x \right] \]

\[ k_{xn} = \begin{cases} 
\sqrt{\frac{\omega^2}{c^2} - (k_2 + m k_0)^2} & |k_2 + m k_0| < \omega/c \\
+i \sqrt{(k_2 + m k_0)^2 - \frac{\omega^2}{c^2}} & |k_2 + m k_0| > \omega/c 
\end{cases} \]
Boundary Condition

\[ e^{ik_z z} \left[ \hat{E}_{\text{inc}} \exp[-ik_x d(z)] + \sum_m \hat{E}_m \exp(ik_0 z + ik_m d(z)) \right] = 0 \]

Specialize to small \( d(z) \)

Specularly reflected wave (\( m = 0 \))

\[ \hat{E}_0 = -\hat{E}_{\text{inc}} \]
OTHER WAVES

\[ \hat{E}_m \left\{ (1-i k_x d)-(1+i k_x d) + \sum_{m} E_m \exp(i m k_0 z) \right\} = 0 \]

\[ E_m = 2i k_x d_m \frac{\hat{E}_{inc}}{\Delta} \]

\[ \frac{\omega \cos \Theta_m}{c} = \sqrt{\frac{\omega^2}{c^2} - \left( \frac{\omega \sin \Theta_m + m k_0}{c} \right)^2} \]

\[ \cos \Theta_m = \sqrt{1 - \left( \frac{\sin \Theta_m + m k_0 c}{\omega} \right)^2} \]
The tape helix model presented here has been incorporated into the large signal TWT simulation code CHRISTINE [17]. The formulation is such that it is easily generalized to the case of a single dielectric layer between the helix and the outer wall. The development of the dispersion relation and power flow for the metal waveguide is treated in the next section.

Tape Helix

![Diagram of a tape helix surrounded by a single dielectric layer enclosed in a cylindrical metal waveguide.](image)

Approximate solution

\[ \omega = kv \]

\[ v_p = c \sqrt{\frac{p}{\sqrt{p^2 + (2\pi r)^2}}} \]

The assumption of an infinitely thin tape presents a question of how the predictions of the theory presented here could be compared with experimental measurements on real, finite tapes. In the usual case, in which the skin depth is taken to be infinitely thin in the radial direction, the narrow tape assumptions are used, as shown in the figure. The developed helix (cut along a plane of constant pitch) is shown in (b). The tape cuts a plane of constant thickness representing the tape thickness, but no attempt to do this is made in the present paper. Here we present the first exact treatment of a single thin tape. Generalizations of the analysis are straightforward.

The electromagnetic waves supported by a tape helix centered inside a perfectly conducting circular cylinder with a single azimuthally uniform dielectric lining, as shown in Fig. 1. The radius of the helix is its width is its period or pitch in the axial direction is its pitch angle of the helix, measured from the vertical plane. The waveguide radius is the metal wall. The interior of the helix is radially stratified into multiple layers with different dielectric constants. This is physically reasonable. In particular, assumption (2) omits the expected square root singularity of the parallel current density at the tape edges, present even for narrow tapes. In the present report it is shown how these common assumptions may be eliminated and a formally exact dispersion relation obtained from which the interaction impedance may be obtained, is also given.

\[ \text{Dielectric region} \]

\[ \text{Metal wall} \]

\[ k \]

\[ v \]

\[ p \]

\[ c \]
Crossings – not gaps

Consequence of helical symmetry

\[ nk_0 \]

\[ k_0 = \frac{2\pi}{p} \]
The matrix longitudinal current on the tape is constant, the transverse current is zero, therefore, the current is a helix with a constant pitch. The interaction impedance is computed from

\[ Z_{\text{pierce}} = \frac{|E_z|^2}{2k^2 P} \]

Pierce: Vacuum electronics pioneer
Pulse code modulation
First communications satellite
Bohlen-Pierce musical scale
Coined name “Transistor”
Higher Dimensions

\[ \nabla^2 E(x, y) + \frac{\omega^2}{c^2} (1 + \chi(x, y)) E(x, y) = 0 \]

\[ \chi(x, y) = \chi(x + d, y) = \chi(x, y + d) \]

\[ E(x, y) = \sum_{m,n} \bar{E}_{m,n} \exp\left[ i \left( k_x + nk_0 \right) x + i \left( k_y + mk_0 \right) \right] \]

\[ \omega(k_x, k_y) = \omega(k_x + qk_0, k_y + pk_0) \]

\[ k_0 = \frac{2\pi}{d} \]
Level curves of frequency in the $k$ plane

$$\omega(k_x, k_y) = \omega(k_x + qk_0, k_y + pk_0)$$

$k_0 = \frac{2\pi}{d}$
Creation of Stop Band

\[
\begin{bmatrix}
\omega^2 - \omega_c^2
\end{bmatrix}
E(n,m) = \frac{\delta}{2} \omega_c^2 \left[ E(n+1,m) + E(n-1,m) + E(n,m+1) + E(n,m-1) \right]
\]

\[
E(n,m) = E(0,0) \exp \left[ i(k_x d n + k_y d m) \right]
\]

\[
\begin{bmatrix}
\omega^2 - \omega_c^2
\end{bmatrix}
= \delta \omega_c^2 \left[ \cos(k_x d) + \cos(k_y d) \right] = \delta \omega_c^2 \cos[(k_x - k_y)d] \cos[(k_x + k_y)d]
\]
\[
[\omega^2 - \omega_c^2]E(n,m) = \frac{\delta}{2} \omega_c^2 \left[ E(n+1,m) + E(n-1,m) + E(n,m+1) + E(n,m-1) \right]
\]

\[
E(n,m) = E(0,0) \exp \left[ i(k_x d_n + k_y d_m) \right]
\]

\[
[\omega^2 - \omega_c^2] = \delta \omega_c^2 \left[ \cos(k_x d) + \cos(k_y d) \right] = \delta \omega_c^2 \cos(k_x d)
\]

Individual cavities have a set of modes, \( \omega_c^2 = \omega_{c_1}^2, \omega_{c_2}^2, \omega_{c_3}^2 \ldots \).

If the spacing between modes is greater than the frequency shift induced by coupling

\[
\left| \omega_{cp} - \omega_{cp+1} \right| < \delta \omega_c^2
\]

then gaps in the spectrum with no propagating modes appear.
Metamaterials

Metamaterials are periodic structures that have engineered properties in the long wave length limit,

\[ kd \ll 1 \]

By Jeffrey.D.Wilson@nasa.gov (Glenn research contact) - NASA Glenn Research, Public Domain, https://commons.wikimedia.org/w/index.php?curid=7455771
Negative epsilon and negative mu

In a restricted range of frequencies the effective constitutive parameters may be negative.

If both are positive or both are negative waves propagate.

\[ k^2 = \omega^2 \varepsilon \mu > 0 \]

If both are negative waves satisfy the left hand rule.

\[ \mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H} \]
For $\varepsilon<0$ or $\mu<0$ they must be functions of frequency.

Media are passive, stored energy is positive.

$$U_E = \frac{1}{2} \frac{\partial}{\partial \omega} (\omega \varepsilon(\omega)) |E|^2 > 0, \quad \frac{\partial}{\partial \omega} (\omega \varepsilon(\omega)) = \varepsilon(\omega) + \omega \frac{\partial}{\partial \omega} \varepsilon(\omega) > 0$$

If both $\varepsilon<0$ and $\mu<0$ group and phase velocities are opposite
If both $\varepsilon<0$ and $\mu<0$ group and phase velocities are opposite

$$\frac{1}{v_g} = \frac{\partial}{\partial \omega} k = \frac{\partial}{\partial \omega} \left( \omega \sqrt{\varepsilon \mu} \right) = \sqrt{\varepsilon \mu} + \frac{\omega}{2\sqrt{\varepsilon \mu}} \frac{\partial}{\partial \omega} (\varepsilon \mu)$$

$$\frac{1}{v_g} = \frac{1}{2\sqrt{\varepsilon \mu}} \left[ \mu \frac{\partial}{\partial \omega} (\omega \varepsilon (\omega)) + \varepsilon \frac{\partial}{\partial \omega} (\omega \mu (\omega)) \right] < 0 \quad \text{if both } \varepsilon \text{ & } \eta < 0$$

Backward Waves