Lecture 20

April 21, 2020
Outline

* Review relativistic classical mechanics

* Lorentz transformation in 4D
  * Invariant separation
  * Proper time

* Four vectors
Short Story

\[ \nabla \times B = J + \frac{\partial E}{\partial t} \quad B = \mu_0 H \]

\[ \nabla \times E = -\frac{\partial B}{\partial t} \quad D = \varepsilon_0 E \]

\[ \nabla \cdot E = \rho/\varepsilon_0 \]

\[ \nabla \cdot B = 0 \]

J = \sum_i q_i \delta(x_i \xi - \xi_0)

p = \int q_i \delta(x_i \xi - \xi_0)

Projective approach

Newton's Eqs

\[ \frac{dp_i}{dt} = q_i (E_i + v_i \times B_i) \quad p_i = m v_i \]

\[ \frac{dx_i}{dt} = v_i \]

What needs to be changed to make things relativistically correct?

\[ p_i \rightarrow m v_i \]

\[ v_i^{\gamma} = \frac{1}{1 - v_i^2/c^2} \]

OK, so long as you stay in one frame.
Lorentz Transform

\[ x' = \gamma (x - vt) \]
\[ t' = \gamma (t - vx/c^2) \]
\[ y' = y \]
\[ z' = z \]

TRANSFORMATION OF
COORDINATES OF AN EVENT

Generalize to arbitrary time

\[ x'_{11} = \gamma \left( x_{11} - \beta \eta \right) \]
\[ \xi t' = \gamma (\xi t - \beta \xi \eta) \]
\[ x'_{12} = x_{12} \]
Four Vectors

\[ \mathbf{A} = (ct, x, y, z) = (A_0, \mathbf{A}) \]

\[ A_0' = \gamma (A_0 - \beta \mathbf{A}) \]

\[ A_i' = \gamma (A_i - \beta A_0) \]

\[ A_0' = \gamma A_0 \]

"Scalar Product" \((A_0, \mathbf{A}) (B_0, \mathbf{B})\)

\[ SP = (A_0 B_0 - A \cdot B) = AOB \]

What is \(SP' = A_0 B_0' - A' \cdot B'\)?

\[ = (A_0 B_0 - A \cdot B) = SP \]
Four Vector Examples

Space-time coordinate \( (ct, z, x, y) \equiv X \)

Space-time wave vector \( \left( \frac{\omega}{c}, k_z, k_x, k_y \right) \equiv K \)

Invariant product
\( K \cdot X = \omega t - k \cdot x = \Phi \) wave phase

Same for all observers
Differentiation

\[
\left( \frac{\partial}{\partial ct}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \text{ is a 3-vector}
\]

Wave Phase \[ \Phi = \omega t - k \cdot x \]

\[ \left( \frac{\partial}{\partial ct}, -\nabla \right) \Phi = \left( \frac{\omega}{c}, k \right) \]

\[ \Phi = \pi n \text{ integer} \]

\[ \text{denotes the crests of the wave} \]

In another frame

\[ \Phi = 2\pi n \]
Four Vectors

This means if \((A_0, A_\mu, A_\perp)\) is a fourvector Field

Then

\[
\frac{2}{\text{det}} A_0 \left( -\frac{2}{\text{det}} \nabla \cdot A \right) \text{ is a Lorentz invariant}
\]

\[
\frac{2}{\text{det}} A_0 + \nabla \cdot A \text{ is a Lorentz invariant}
\]

Conservation of charge

\[
\frac{2}{\text{det}} \rho + \nabla \cdot J = 0
\]

\[
\frac{2}{\text{det}} c \rho + \nabla \cdot J = 0 \quad (c \rho, J)
\]

is a 4-vector.
Addition of velocities

\[ u = (u_1, u_2, u_3) \] (not part of 4-vector

\[ u'_\mu = \frac{u_\mu - V}{\sqrt{1 - \frac{V^2}{c^2}}} \quad \Rightarrow \quad u_\mu \text{ refer to } \beta \]

\[ u'_\mu = \frac{u_\mu}{\sqrt{1 - \frac{V^2}{c^2}}} \]
Force on a charge $Q$ moving with speed $v$ in z-direction

$$F = -Qv \times E = -Qv B_0 \hat{z} = -\frac{Qv^2}{2m_e} \frac{e}{m_e}$$
Switch to moving frame

\[ \text{Electrons} \]
\[ \begin{align*}
\dot{c}p_e' &= \gamma (c_p - \beta J_e) \\
\dot{J}_e &= \gamma (\dot{J}_e - \beta c_p) \\
\dot{c}p_e &= \gamma (c_p - \beta J_e) \\
\dot{J}_e' &= 0 \\
\end{align*} \]

\[ \text{Ions} \]
\[ \begin{align*}
\dot{c}p_i' &= \gamma (c_p - \beta J_i) \\
\dot{J}_i &= \gamma (\dot{J}_i - \beta c_p) \\
\dot{c}p_i &= \gamma (c_p - \beta J_i) \\
\dot{J}_i' &= 0 \\
\end{align*} \]
Electric Field in Primed Frame

\[ E'_r = \frac{1}{2\pi \varepsilon_0 r} \pi a^2 \left( \rho'_r + \rho'_e \right) \]

\[ E'_r = \frac{1}{2\pi \varepsilon_0 r} \pi a^2 (\gamma \rho'_r + \frac{1}{\gamma} \rho'_e) \]

\[ = \frac{1}{2\pi \varepsilon_0 r} \pi a^2 \rho'_e \left( \frac{1}{\gamma} - \gamma \right) \]

\[ = \frac{1}{2\pi \varepsilon_0 r} \gamma \pi a^2 \rho'_e \left( -\frac{V^2}{c^2} \right) - \frac{1}{\gamma} - 1 = -\frac{V^2}{c^2} \]

\[ F'_r = -\frac{\partial \rho'_r}{\partial t} - \frac{\partial \rho'_e}{\partial t} \frac{\pi a^2 \rho'_e v^2}{2\pi \varepsilon_0 r} \]

\[ = -\gamma \frac{Qv^2 \gamma_0 v^2}{2\pi \varepsilon_0 r} \]

\[ \text{Account for proper} \]

\[ d^\gamma = \frac{dt}{\delta} \]

\[ \frac{d\rho'_r}{dt} = \frac{d\gamma}{d\tau} \frac{d\rho'_r}{d\tau} \]

\[ \frac{1}{\delta} \]
Transformation of Maxwell’s Equations

\[
\left( \frac{2}{c^2 \frac{\partial^2}{\partial t^2}} - \nabla^2 \right) \phi = \frac{\partial E}{\partial t}
\]

and

\[
\left( \frac{2}{c^2 \frac{\partial^2}{\partial t^2}} - \nabla^2 \right) A = \frac{\partial B}{\partial t}
\]

Thus, \((\phi, A)\) is a 4-vector.

\[
\left( \frac{1}{c^2 \frac{\partial^2}{\partial t^2}} - \nabla^2 \right) A = \mu_0 \mathbf{j}
\]

\[
\left( \frac{1}{c^2 \frac{\partial^2}{\partial t^2}} - \nabla^2 \right) \phi = \frac{\mathbf{R} \cdot \mathbf{E}}{\varepsilon_0 c^2} \quad \varepsilon_0 c^2 = \frac{4 \pi \mu_0}{c^2}
\]

\[
= \frac{c \rho}{(60 c^2)} = \mu_0 (c \rho)
\]

The same equation in all frames.
Relation between Energy and momentum

and proper time

consider the trajectory of a particle defined by the 4-vector

\[(ct, x(t))\]

The proper time for this particle is defined by the equation

\[d\gamma = \frac{dt}{\gamma(u(t))} \quad u(t) = \frac{dx(t)}{dt}\]
differentiate the position four vector

\( (ct, x(t)) \) with respect to

proper time

\[
\frac{d (ct, x(t))}{d \tau} = \left( c \frac{dt}{d\tau}, \frac{dx(t)}{d\tau} \right)
\]

also a four vector, why

\( (c dt, dx(t)) \) is a four vector

\( \frac{1}{d\tau} \) is a Lorentz invariant

since is calculated in any frame

thus their ratio is a four vector
Transformation of fields

\[ (\Phi, A) \leftrightarrow (\Phi', A') \]

\[ E'_\perp = \gamma (E'_\perp + v \times B'_\perp) \]

\[ B'_\perp = \gamma \left( B'_\perp - \frac{v \times E'}{c^2} \right) \]

\[ E'_\parallel = E_\parallel \]

\[ B'_\parallel = B_\parallel \]
Relativistic Energy

The **total energy** $E$ of a particle is

$$E = \gamma_p mc^2 = E_0 + K = \text{rest energy} + \text{kinetic energy}$$

This total energy consists of a **rest energy**

$$E_0 = mc^2$$

and a relativistic expression for the **kinetic energy**

$$K = (\gamma_p - 1)mc^2 = (\gamma_p - 1)E_0$$

This expression for the kinetic energy is very nearly $mu^2/2$ when $u << c$. 
Where does this definition of energy come from?

\[ \frac{d}{dt} \gamma_p mc^2 = mc^2 \frac{d}{dt} \sqrt{1 + (p / mc)^2} = \frac{p}{m\sqrt{1 + (p / mc)^2}} \frac{dp}{dt} \]

Thus,

\[ \frac{d}{dt} \gamma_p mc^2 = \frac{p}{m\gamma_p} \frac{dp}{dt} = u \frac{dp}{dt} = uF \]

Rate at which work is done

Replaces kinetic energy
Energy of EM waves and particles now given by the same formula

Energy in an EM wave

\[ E = pc \]

p = momentum in an EM wave

For particles:

\[ E = \gamma_p mc^2 = mc^2 \sqrt{1 + (p / mc)^2} = c \sqrt{(mc)^2 + (p)^2} \]

Let \( m \rightarrow 0 \) \quad \quad E \rightarrow pc \]
Revised Newton’s Laws

\#1 \[ \frac{d}{dt} \mathbf{p}_i = q \left( \mathbf{E} + \mathbf{v}_i \times \mathbf{B} \right) \]

\#2 \[ \mathbf{p}_i = m \gamma_i \mathbf{v}_i \]
\[ \gamma_i = \frac{1}{\sqrt{1 - \left| \mathbf{v}_i \right|^2 / c^2}} \]

Momentum can become large, but particle speed is always less than c

\#3 \[ \frac{d}{dt} \mathbf{x}_i = \mathbf{v}_i \]
Relativistic Energy

The total energy $E$ of a particle is

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This total energy consists of a rest energy

$$E_0 = mc^2$$

and a relativistic expression for the kinetic energy

$$K = (\gamma_p - 1)mc^2 = (\gamma_p - 1)E_0$$

This expression for the kinetic energy is very nearly $mu^2/2$ when $u \ll c$. 
Energy

\[ E = \gamma_p m c^2 \]

Velocity

\[ \gamma_p = 1 / \sqrt{1 - u^2 / c^2} \]

\[ \vec{u} = \vec{p} / (m \gamma_p) \]

Momentum

\[ \gamma_p = \sqrt{1 + (p / mc)^2} \]

\[ p = mc \sqrt{\gamma_p^2 - 1} \]

\[ u = c \sqrt{1 - 1 / \gamma_p^2} \]
Mass Energy Equivalence

Isolated box of mass $M$ and length $L$ in space. A light on the wall on one side sends out a photon of energy $E$ toward the right. The photon has momentum $p=E/c$. The box recoils with velocity $v=p/M$ to the left. The photon is absorbed on the other side after a time $T=L/c$. The box absorbs the momentum and stops moving.

Displacement of the box

$$\Delta x = vT = \frac{EL}{Mc^2}$$

Has the center of mass moved? We would like to say no. The box shouldn’t be able to move its center of mass.

We can say that the CM hasn’t moved if the photon reduced the mass of the left side by $m=E/c^2$ and increased the right side by the same amount.

$$E = mc^2$$
Relativistic Charged Particle Motion

Relativistic Momentum Equation (esu)
\[
\frac{d}{dt} \mathbf{p} = q \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) = q \left( -\nabla \Phi - \frac{\partial}{\partial t} \mathbf{A} + \frac{\mathbf{v} \times \nabla \times \mathbf{A}}{c} \right)
\]

\[\mathbf{v} \times \nabla \times \mathbf{A} = \nabla (\mathbf{v} \cdot \mathbf{A}) - \mathbf{v} \cdot \nabla \mathbf{A}\]  
Vector identity

\[
\frac{d}{dt} \mathbf{p} = q \left( -\frac{1}{c} \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{A} - \nabla \left( \Phi - \frac{\mathbf{v} \cdot \mathbf{A}}{c} \right) \right)
\]

\[
\frac{d}{dt} \mathbf{P} = -q \nabla \left( \Phi - \frac{\mathbf{v} \cdot \mathbf{A}}{c} \right)
\]

\[\mathbf{P} = \mathbf{p} + \frac{q}{c} \mathbf{A}\]  
Canonical Momentum
Hamiltonian - Energy

\[
H(P, x, t) = mc^2 \gamma + q \Phi
\]

\[
\gamma = \left(1 + \frac{P^2}{m^2 c^2}\right)^{1/2}
\]

Write relativistic factor in terms of \( P \)

Differentiate \( H \) w.r.t. time

\[
\frac{d}{dt} H = \frac{1}{m \gamma} \left( P - qA / c \right) \cdot \frac{d}{dt} \left( P - qA / c \right) + q \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \Phi
\]

\[
\frac{d}{dt} H = v \cdot \frac{d}{dt} p + q \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \Phi = q v \cdot E + q \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \Phi = \frac{\partial}{\partial t} H
\]

Time independent \( H \) is constant
Conservation Laws

\[
\frac{d}{dt} P = -q \nabla \left( \Phi - \frac{\mathbf{v} \cdot \mathbf{A}}{c} \right) = -\nabla H
\]

\[
\frac{d}{dt} P_z = -q \frac{\partial}{\partial z} \left( \Phi - \frac{\mathbf{v} \cdot \mathbf{A}}{c} \right) = -\frac{\partial H}{\partial z}
\]

If fields only depend on z-ct (plane wave) then:

\[
\frac{dH}{dt} = \frac{\partial H}{\partial t}
\]

\[
P_x = \text{constant}
\]

\[
P_y = \text{constant}
\]

\[
H - cP_z = \text{constant}
\]
• Plane Wave Laser Field: \[ E = -\frac{\partial}{c\partial t} A_\perp = \left[ i\frac{\omega}{c} \hat{A}_\perp \exp[-i\omega(t-z/c)] + c.c. \right]/2 \]

• Electron Hamiltonian: \[ H = mc^2 \gamma = H(P, t - z/c), \quad P = p + \frac{qA}{c} \]

• Relativistic Factor: \[ \gamma = \sqrt{1 + \left( \frac{P_z}{mc} \right)^2 + \left( \frac{P_\perp - qA_\perp}{mc} \right)^2} \]

• Hamilton's Equations:

\[ \frac{dP_\perp}{dt} = -\frac{\partial H}{\partial x_\perp} \quad \Rightarrow \ P_\perp = \text{constant} \]

\[ \frac{dH}{dt} = \frac{\partial H}{\partial t}, \quad \frac{dp_z}{dt} = -\frac{\partial H}{\partial z} \quad \Rightarrow \ H - cP_z = \text{constant} \]
- **Laser Field:**
  \[ E = -\frac{1}{c} \frac{\partial}{\partial t} A(x_\perp, z, t) \]
  Weak dependence on transverse coordinate

- **Transverse Canonical Momentum:**
  \[ p_\perp + \frac{q}{c} A_\perp = \text{const.} = 0 \]

- **Quiver velocity:**
  \[ \frac{p_\perp}{mc} = \frac{\gamma v_\perp}{c} = -\frac{qA_\perp}{mc^2} = -a \]
  Normalized vector potential

- **Example:**
  \[ I = 10^{18} \text{ watts/cm}^2, \quad \lambda = 10^{-4} \text{ cm}, \quad |a| = .86 \]
Assume electrons originate in field-free region: \( p_{\perp} = 0, \quad p_{z0} = 0, \quad \gamma_0 = 1 \)

Constants of motion imply: \[ p_{\perp} = -\frac{qA_{\perp}}{c} \propto -\cos \omega t, \quad p_{z} = \frac{p_{\perp}^{2}}{2mc} \propto \cos^2 \omega t = \frac{1+\cos 2\omega t}{2} \]

Mean drift in propagation direction

Figure-8 in drifting frame
- **Ponderomotive Force:** low frequency force, quadratic in field strength

- **Lorentz force:** 
  \[ F = q(E(x,t) + \frac{v \times B(x,t)}{c}) \]

- **Trajectory:** 
  \[ x(t) = x_0(t) + \ddot{x}(t) \]
  Rapid quiver
  Slowly varying

- **Slowly varying component:** 
  \[ F_p = q \left( \ddot{x} \cdot \nabla E(x_0,t) + \frac{\ddot{v} \times B(x_0,t)}{c} \right) \text{ Laser period} \]

- **Ponderomotive Potential:** 
  \[ F_p = -\frac{mc^2}{2\gamma} \nabla \langle |a|^2 \rangle \text{ Proportional to laser intensity} \]