Lecture 3

Inductance  Self/Mutual
Displacement  Current
Scalar and Vector Potentials

F  Surfaced Potential

Components
Inductance

\[ \sum E \cdot dl = -N \frac{d\Phi}{dt} = -N \frac{d}{dt} \left( \frac{N I^2 r}{L} \right) \]

\[ V(1) - V(2) = -\frac{1}{2} L \frac{dI}{dt} \]

\[ V_{12} = -I \frac{d}{dt} \]

Inductance - mho

\[ \left( \begin{array}{c} V_1 \\ V_2 \end{array} \right) = \left( \begin{array}{cc} L_1 & L_{12} \\ L_{12} & L_2 \end{array} \right) \left( \begin{array}{c} I_1 \\ I_2 \end{array} \right) \]

Debye mho

\[ k = \frac{L_2}{V_{\text{meter}}} \]

\[ \text{permittivity} \]
Inductance Matrix

\[
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix} =
\begin{bmatrix}
L_{11} & L_{12} \\
L_{21} & L_{22}
\end{bmatrix}
\begin{pmatrix}
di/dt \\
di/dt
\end{pmatrix}
\]

\[d\ V = L \frac{di}{dt}\]

Properties: \(L_{12} = L_{21}\) reciprocal \(L_{11}, L_{22} > 0\)

\[\text{Det}(L) > 0\] eigen

Power in = \(V_1 I_1 + V_2 I_2\) = \(\frac{1}{2} I^T \frac{di}{dt}\)

\[I^T \frac{di}{dt} = I^T i\]

Energy stored = \(\frac{1}{2} I^T L I\)
Energy stored

\[ V = \int T \, \text{d}T \]

\[ V_i = \int T_i \, \text{d}T_i \]

\[ U = \int_{-\infty}^{t} \sum_{i} \left( \frac{d}{dt} L_{ij} \right) u_j \, \text{d}t \]

If \( \det(L_{ij}) < 0 \) of eigenvalue,

Then there is an eigenvalue \( \lambda < 0 \) \( \lambda_j = e \)

\[ I_j = I u_j \]

\[ \sum_{j} L_{ij} u_i = \lambda u_i \]

\[ U = \int_{-\infty}^{t} I(t) \, \frac{d}{dt} \sum_{j} u_j^2 \]
Two Loops

$\phi_{21} \approx 0$
$\phi_{32} \approx 0$
$L_{12} \geq 0$ ? $L_{12} < 0$

$L_{12} \geq 0$ ? $L_{12} > 0$

$L_{12} \approx 0$

$\omega_{\text{metae}}$
MAXWELL DETERMINED THAT AMPERE'S LAW

4) \[ \int_{C} H \cdot dL = \int_{S} \mathbf{J} \cdot dS \]

could not be correct if fields changed in time. 4) is inconsistent with conservation of charge.

CONSIDER THE FOLLOWING EXAMPLE

\[ Q(t) \text{ charge on plate} \]

\[ \oint_{S} \mathbf{J} \cdot dS = \frac{d}{dt} \left[ \int_{S_1} \mathbf{E} \cdot dA + \int_{S_2} \mathbf{E} \cdot dA \right] = I \]

\[ Q(t) \text{ increases with time} \]
apply Ampere's Law to C

\[ \oint_{C} \mathbf{H} \cdot d\mathbf{l} = \int_{S_{1}} d\mathbf{s} \cdot \mathbf{j} \]

For \( S_{1} \), \[ \int_{S_{1}} d\mathbf{s} \cdot \mathbf{j} = I \]

For \( S_{2} \), \[ \int_{S_{2}} d\mathbf{s} \cdot \mathbf{j} = 0 \]

**SOMETHING IS WRONG ANSWER SHOULD NOT DEPEND ON CHOICE OF SURFACE.**

Remember Far

THE PROBLEM IS THAT IN CASE OF FARADAY'S LAW

\[ \oint_{C} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{S} \mathbf{B} \cdot d\mathbf{s} \]

\[ \int_{S} \mathbf{B} \cdot d\mathbf{s} = 0 \quad \text{integral above independent of closed surface} \]

makes a choice of \( S \).
However, conservation of charge states

\[ \int_{\text{closed surface}} J \cdot ds = -\frac{\partial Q}{\partial t} \]

where \( Q(t) \) = charge enclosed by surface

\[ \int_{\text{closed surface}} J \cdot ds = 0 \quad \text{for static fields} \ (\frac{\partial \mathbf{E}}{\partial t} = 0) \]

Ampere's law is not for non-statics.

How can we "fix" Ampere's law?

\[ Q = \int_{\text{closed surface}} \mathbf{D} \cdot ds \]

Poisson's equation

Conservation of charge implies

\[ \int_{\text{closed}} ds \cdot (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) = 0 \]
THUS AMPEDE'S LAW SHOULD BE MODIFIED AS FOLLOWS

\[ \int_{C} H \cdot dL = \int_{S} ds \cdot (j + \frac{\partial D}{\partial t}) \]

on \( S_1 \) \( (\rho = 0) \) \( \int_{S_1} ds \cdot (j + \frac{\partial D}{\partial t}) = I \)

on \( S_2 \)

\[ \int_{S_2} ds \cdot (j + \frac{\partial D}{\partial t}) \approx A \frac{\partial}{\partial t} \left( \frac{\rho}{A} \right) \]

\[ j = 0 \]

\[ \frac{\partial \rho}{\partial t} = I \]

IT WORKS
Conduction Current vs. Displacement Current

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

Suppose we are in vacuum \( \vec{D} = \varepsilon_0 \vec{E} \)

\( \vec{J} \) is the current resulting from the movement of charge, what you normally think of as current

\[ \frac{\partial \varepsilon_0 \vec{E}}{\partial t} = \text{Maxwell's displacement current} \]

Time changing \( \vec{E} \) induces \( \vec{H} \)
suppose we now consider

a dielectric medium

\[ P = P_{\text{free}} + P_{\text{induced}} \]

\[ \Delta \]

charge density

induced in the

medium by stretching

and alignment of

molecules

\[ E = 0 \]

\( \text{neutral} \)

\( \text{molecule} \)

\[ E \neq 0 \]

molecules become "polarized"

E changed

and charge moved
\[ \vec{P}_{\text{induced}} = - \nabla \cdot \vec{P} \]

differentiate with time

\[ \frac{\partial \vec{P}_{\text{induced}}}{\partial t} = - \nabla \cdot \frac{\partial \vec{P}}{\partial t} \]

Thus, \( \frac{\partial \vec{P}}{\partial t} \) is the current associated with the polarization of the medium.

\[ \vec{J} = \vec{J}_{\text{free}} + \frac{\partial \vec{P}}{\partial t} \]

\[ \vec{J} = \vec{J}_{\text{free}} + \frac{\partial \vec{P}}{\partial t} + \frac{\partial}{\partial t} \varepsilon_0 \vec{E} \]

\[ = \vec{J}_{\text{free}} + \frac{\partial}{\partial t} \vec{P} \]

In a dielectric medium, the term \( \frac{\partial \vec{P}}{\partial t} \) includes both Maxwell stress and polarization current.
Problem 4.2
Find field outside

Start with the assumption that fields are electrostatic

\[ \mathbf{E}_z = \frac{Q}{\epsilon_0} \cdot \mathbf{a}_z \]

\[ \mathbf{E}_{z} = -\frac{d\phi}{dt} \]

Ampere's Law with Displacement Current

\[ (\nabla \times \mathbf{E})_z = \mu_0 \epsilon_0 \frac{d\mathbf{E}_z}{dt} \]

\[ \frac{1}{2} \frac{d}{dt} \mathbf{B}_0 = \mu_0 \epsilon_0 \frac{d\mathbf{E}_z}{dt} \]

Faraday's Law

\[ -\frac{dB_y}{dt} = (\nabla \times \mathbf{E})_y = -\frac{d}{dt} \mathbf{E}_y \]

\[ E_2(\tau) = E_3(0) + \int_0^\tau \frac{1}{2} \frac{d}{dt} \left( \frac{\mu_0 r^2}{\epsilon_0} \right) \mathbf{E}_z \cdot \mathbf{E}_z = E_3(0) + \frac{\mu_0 r^2}{4 \epsilon_0} \frac{d^2 E_3(0)}{dt^2} \]

Suppose \( E_2(\omega) = E_0 \sin \omega t \)

\[ A = \frac{C_1}{\sqrt{2}} \]

Require \( \frac{\mu_0 r^2 v^2}{4 \epsilon_0} \ll 1 \)
Actually, we should solve

\[
\frac{1}{2} \bar{E}_0 B_0 = \frac{\varepsilon_0 \mu_0}{\varepsilon}\frac{\partial E^2}{\partial t}
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E^2}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 E^2}{\partial t^2} - \frac{2}{c^2} E^2
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial E^2}{\partial r} \right) - \frac{1}{c^2} \frac{\partial^2 E^2}{\partial t^2} = 0
\]

\[
E_2 = \text{Im} \left\{ E_0 \text{e}^{-j \omega t} \right\}
\]

Bessel's Eqn. \hspace{1cm} (8.7)

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_2}{\partial r} \right) + \frac{\omega^2}{c^2} E_2 = 0
\]

Bessel's Eqn.

\[
\tilde{E}_2 = E_0 J_0 (\omega r) \hspace{1cm} J_0 (x) = 1 - \frac{1}{4} x^2 + 0 (x^4)
\]

\[
J_0 (x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{(m!)^2}
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial J_0 + J_0}{} \right) = 0
\]
Scalar and Vector Potentials

\[ \nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A} \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{c^2} \nabla \times \mathbf{A} \]

\[ \nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0 \]

\[ \Rightarrow \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \]

\[ \mathbf{E} = -\left( \frac{\partial \mathbf{A}}{\partial t} + \mathbf{A} \times \nabla \Phi \right) \]

Putting these into (vacuum: \( \varepsilon = \varepsilon_0 \), \( \mu = \mu_0 \))

\[ \nabla \cdot \mathbf{E} = \rho \varepsilon_0 \]

and

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad \frac{1}{\varepsilon_0} = \mu_0 \varepsilon_0 \]

gives

\[ \nabla^2 \Phi + \frac{\partial^2}{\partial t^2} (\nabla \cdot \mathbf{A}) = -\frac{4\pi}{\varepsilon_0} \]

\[ \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} + \frac{1}{c^2} \left( \frac{\partial \mathbf{A}}{\partial t} + \nabla \Phi \right) \]

\[ \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\frac{\varepsilon_0}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \mathbf{J} \]

**Note:** Eqs. for \( \Phi \) and \( \mathbf{A} \) are coupled. **Nasty**
\( A \) and \( \Phi \) have an arbitrary \( \eta \):

\[
\begin{align*}
A &\rightarrow A' = A + \nabla \eta \\
\Phi &\rightarrow \Phi' = \Phi - \frac{\partial \eta}{\partial t}
\end{align*}
\]

This arbitrary \( \eta \) is a reflection that we can choose \( \nabla \cdot A \) to be whatever we want.

Lorentz gauge:

\[
\nabla \cdot A + \frac{\partial \Phi}{\partial t} = 0
\]

With this gauge the equations become:

**Vector wave eq.**

\[
\nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\mu_0 J
\]

**Scalar wave eq.**

\[
\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\rho_0}
\]

\( \Delta \) NOTE: Eqs. for \( A \) & \( \Phi \) decouple.

\( A \), \( \Phi \) is not in the Lorentz gauge

\[
\nabla \cdot (A + \frac{\partial \Phi}{\partial t}) \neq 0
\]

Can we do a gauge transformation to make it so?
Let $A' = A + \nabla \Lambda$

$\Phi' = \Phi - \frac{c}{a} \frac{\partial A}{\partial t}$

Then

$\nabla \cdot A' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} = \left( \nabla \cdot A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) + \left( \nabla^2 \Lambda = \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} \right)$

So $\nabla \cdot A' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} = 0$ if we can choose $\Lambda$ so that

the $\nabla \cdot A' = 0$ ⇒

$\nabla^2 \Lambda = \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\left( \nabla \cdot A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right)$

This equation (on we will

This equation is inhomogeneous wave equation is solvable for $A$. 

$A', \Phi'$ can be chosen to satisfy the

Lorentz gauge.

Note: There are many potentials that satisfy the

Lorentz gauge.

From above, if $(A, \Phi)$ satisfies the Lorentz gauge, then so does $(A', \Phi')$ provided that

$\nabla^2 A' = \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0$. 