Lecture #2

- Faraday's Law
- Many logs
- Skin Effect
MODIFICATIONS FOR DYNAMIC FIELDS

3) $\oint_{C} E \cdot dl = -\frac{2}{\mu_{0}} \int_{S} B \cdot ds$  Faraday's Law
   \[ \nabla \times E = \frac{2}{\mu_{0}} \frac{d}{dt} B \]

4) $\oint_{C} H \cdot dl = \int_{S} J \cdot ds + \frac{2}{\epsilon_{0}} \int_{S} \mathcal{D} \cdot ds$
   Maxwell's displacement current
   \[ \nabla \times H = \frac{2}{\epsilon_{0}} \frac{d}{dt} E \]

3) Faraday's Law determined experimentally

![Diagram of a wire loop with a volt meter connected](attachment:diagram.png)
As the magnet was moved the voltage appeared at the meter. The sign of the voltage depended on whether the magnetic flux threading the loop was increasing with time or decreasing. If the magnet was held still \( \frac{\partial B}{\partial t} = 0 \) the voltage was zero.

Experimentally it was found

\[
V = - \int_{L_c} E \cdot dl = \frac{d\psi}{dt}
\]

where \( \psi = \int_{S_{\text{loop}}} B \cdot ds \)

For stationary loops
SIGN DETERMINED BY THE
FOLLOWING CONVENTION

RIGHT HAND RULE

direction of $dl$ determines
direction of $ds$

\[-\int_{C} E \cdot dl = \frac{d}{dt} \int_{S} B \cdot ds\]

ALTERNATE METHOD: LENS' LAW

IF THE LOOP were a conducting wire
sign of electric field is such that
the resulting current would oppose
the change in $B$. 
do not confuse $\int_S \mathbf{B} \cdot d\mathbf{s} = 0$ with $\Psi$

Gauss' Law tells us

$\frac{\int_{S_1+S_2} \mathbf{d}s \cdot \mathbf{B}}{S_1+S_2} = 0$

Faraday's Law works for any surface whose perimeter is the curve $\mathbf{C}$
Consider two surfaces $S_1$ and $S_2$ as shown.

$$\psi_1 = \int_{S_1} ds \cdot B \quad \psi_2 = \int_{S_2} ds \cdot B$$

Does $\psi_1 = \psi_2$? 

Answer: yes

The two surfaces together make a closed surface.

Outward normal $ds = ds_1$ on $S_1$

$ds = -ds_2$ on $S_2$

Gauss' Law

$$\int_{S_1} ds \cdot B + \int_{S_2} ds \cdot B = 0$$

Outward normal

$$\psi_1 - \psi_2 = 0$$

Thus $\psi_1 = \psi_2$

Faraday's law holds for any surface.
Using Stokes' Theorem

\[ \oint_S \mathbf{E} \cdot d\mathbf{L} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \]

True for any stationary loop and corresponding surface S.

Thus,

\[ \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \]

Faraday's Law in differential form.
EMF around a moving rectangular loop (Three cases)

Suppose there is an applied magnetic field of the form

\[ B(x,y,z,t) = \begin{cases} 
B_0 \hat{z} & \text{for } x > 0 \\
0 & \text{for } x < 0 
\end{cases} \]

and a rectangular loop moving in the +x direction (with speed \( v \)) as shown.

What is the EMF around the loop?

Case #1: The loop is not a conductor, or at least not a very good conductor. Thus, no current is carried by the loop, and as a result, the magnetic field is given by the applied magnetic field. (If a significant current is carried by the loop, it can modify the magnetic field as we will see in Case 3.)
Pick \( dl \) as shown in the figure. The flux passing through the loop is confined to the cross-hatched region and is given by
\[
\psi(t) = \int_{S} dxdy \int_{A(t)} \hat{z} \cdot B = \frac{\hbar \cdot \nu t - B_0}{A(t)}
\]

\( A(t) \) is the area of the cross-hatched region. You can verify that for the chosen direction of \( dl \), \( d\mathbf{s} = \hat{z} dxdy \).

\[
V = \text{induced emf} = -\frac{d\psi}{dt} = -\hbar \nu B_0
\]

An alternate way of calculating the emf is to do the line integral
\[
V = \oint_{C} \mathbf{E} \cdot d\mathbf{l} + \oint_{C} \mathbf{E} \times d\mathbf{B}
\]
around the loop. For this problem, \( \partial B / \partial t = 0 \), and there is no electric charge present, so
\[
\mathbf{E} = 0.
\]

Thus, all the contribution comes from the so-called motional emf, \( \oint_{C} \mathbf{E} \times d\mathbf{B} \). To calculate this we note
\[
\mathbf{E} \times \mathbf{B} = -\nu \mathbf{B} \cdot \mathbf{F}.
\]
Thus, all the contribution comes from the end of the wire whose length is h. Note that \( dL = \hat{y} dy \) on this end. Thus

\[
\text{EMF} = \mathcal{E} = \oint_{C} \mathbf{V} \cdot \mathbf{B} = \int_{0}^{h} dy \hat{y} \cdot (-v \mathbf{B}_{0} \hat{y}) = -hv \mathbf{F}
\]

which agrees with \( -d\psi/dt \) as it should.

Case #2: The loop is a good conductor but the end is an open circuit as shown.

Because of the open circuit, no current can flow around the loop. Again, the magnetic field is given by the applied magnetic field. As a result, the EMF is the same as before.

\[
\mathcal{E} = \text{induced EMF} = -\frac{d\psi}{dt} = -hv \mathbf{B}_{0}
\]
This must also be the same as the line integral $\int_{\partial L} (E + V \times B)$, but the picture is somewhat changed due to the moving conductor. Electrons in the end of the moving conductor feel the Lorentz force $-1e1V \times B = 1e1VB_0 \hat{j}$. This causes them to move upward and accumulate at the top of the loop.

The resulting charge imbalance produces an electrostatic field $E$. Electrons adjust their positions until

$$E + V \times B = 0$$

in the conductor. (Remember it is a good conductor.) The above shows that

$$E = E_y \hat{y} = -V \times B = -(VB_0 \hat{j}) = VB_0 \hat{j}$$
Now let's calculate $V = \oint (E + \nabla \times B) \cdot dl$ around the loop. Since $E + \nabla \times B = 0$ in the conducting wire, all the contribution to $V$ comes from the gap (where $B = 0$).

$V = -\int_{\text{gap}} dl \cdot E = -\int_{\text{gap}} dy \cdot Ey$

We must still make a complete loop even though there is a gap in the circuit.

How do we known that $V$ calculated this way still gives $-\partial \Psi / \partial t$? Separate the electric and motional contributions to $V$.

$V = \oint_{\text{loop}} (E + \nabla \times B) = \oint_{\text{loop}} dl \cdot E + \oint_{\text{loop}} dl \cdot \nabla \times B$

Since the electric field is electrostatic, $\oint_{\text{loop}} dl \cdot E = 0$. The case a contribution $\int_{\text{gap}} dy \cdot Ey$ from the right end exactly cancels the contribution from the gap. The motional contribution is the same as in case #1.
Thus it is still true that

\[ \frac{d\psi}{dt} - \oint dL \cdot (E + V \times B) = h \nu B_0. \]

**Case #3** The gap in case #2 is closed. Now current can flow in the loop. The direction of current flow is shown below.

The flowing current produces a self magnetic field which opposes the changing flux of the applied field. The result is

\[ \frac{d\psi}{dt} = \frac{d}{dt} \int d\mathbf{s} \cdot \mathbf{B} = -\oint d\mathbf{L} \cdot (E + V \times B) = 0. \]

\[ \mathbf{B} = B_{\text{applied}} + B_{\text{self}} \int d\mathbf{s}, B_{\text{applied}} = -\int d\mathbf{s}, B_{\text{self}}. \]

(Self flux cancels applied flux.)
Example: Time varying field

find the electric field in the gap.

assume fields are azimuthally symmetric
VIEWED FROM ABOVE

B out of page

← magnet

take a loop of radius \( r \)

by symmetry \( \frac{\partial E_0}{\partial \theta} = 0 \) \( E_0(r) \)

\[ \oint_{C} d\mathbf{l} \cdot E_0 = \text{area} E_0(r) \]

\[ \int_{S} d\mathbf{s} \cdot \mathbf{B} = \int_{0}^{2\pi} \int_{0}^{r} r dr d\theta B_z(r) \]

assume \( B_z = B_0(t) \) for \( r < a \)

\( B_z = 0 \) for \( r > a \)
\[ \psi = \pi r^2 B_0 \quad r < a \]

\[ \psi = \pi a^2 B_0 \quad r > a \]

\[ 2 \pi r E_\theta = -\frac{d\psi}{dt} = - \pi \frac{dB_0}{dt} \left\{ \begin{array}{ll} r^2 & r < a \\ a^2 & r > a \end{array} \right. \]

\[ E_\theta = - \frac{1}{2} \frac{dB_0}{dt} \left\{ \begin{array}{ll} \frac{r}{a} & r < a \\ \frac{a^2}{r} & r > a \end{array} \right. \]

\[ \frac{dB_0}{dt} \]

[Diagram of \( E_\theta \) vs. \( r \)]
Moving loops

So far we have discussed only stationary loops and surfaces.

Suppose we have a moving loop.

\[ \vec{X}(\vec{x}, t) \]

Each point of the loop is moving with velocity \( \vec{V}(\vec{x}, t) \).

Then we can show

\[
\frac{d\psi}{dt} = \oint_{C(t)} \frac{d\vec{B}}{dt} \cdot d\vec{s} - \oint_{C(t)} d\vec{l} \cdot \vec{V} \times \vec{B}
\]

\[ \text{Contribution from changing } \vec{B} \quad \text{from changing shape} \]

\[ \Delta A = |\vec{V} \Delta t| \tilde{d} \cdot \sin \theta = |\vec{V} \times d\tilde{l}| \Delta t \]

\[ \text{Contribution from changing shape} \]
\[ \frac{d\mathbf{B}}{dt} = -\nabla \times \mathbf{E} \]

By Stokes' Law

\[ \oint \frac{d\mathbf{B}}{dt} \cdot ds = -\oint \mathbf{E} \cdot \nabla \times \mathbf{E} = -\oint_{c(t)} \mathbf{E} \cdot d\mathbf{l} \]

Thus for moving loops

\[ \gamma' = -\frac{d}{dt} \Phi \]

\[ \frac{d\psi}{dt} = -\oint_{c(t)} \mathbf{E} \cdot (\mathbf{E} + \gamma \times \mathbf{B}) - \]

**ELECTRO MOTIVE FORCE**

\[ \mathcal{E}_{\text{emf}} = \gamma' \oint_{c(t)} \mathbf{E} \cdot (\mathbf{E} + \gamma \times \mathbf{B}) \]

Two ways to compute emf

1. Suppose the moving loop was a wire

\[ \mathbf{F} = q \left( \mathbf{E} + \gamma \times \mathbf{B} \right) \]
Example

$B_z(t)$ constant in space
But varying in time
$R(t)$ varies in time

Area

$\psi(t) = \pi R(t)^2 B_z(t)$

\[
\frac{d\psi(t)}{dt} = \pi R(t)^2 \frac{dB_z}{dt} + \pi R \frac{dR}{dt} B_z
\]

\[
\int_{s(t)} d\mathbf{s} \cdot \frac{\partial \mathbf{B}_z}{\partial t} = \oint_{c(t)} \mathbf{d}l \cdot \mathbf{v} \times \mathbf{B}
\]

Contribution from changing $B_z$

$\mathbf{v} = \partial_{R} \frac{dR}{dt}$
Skin Effect

\[ B = (0, 0, B_z) \]
\[ E = (0, E_y, 0) \]
\[ J = (0, J_y, 0) \]
\[ \frac{\partial E_y}{\partial t} = -\frac{1}{\mu_0 \varepsilon_0} \frac{\partial B_z}{\partial x} \]
\[ \frac{\partial E_y}{\partial x} = -\frac{1}{\mu_0 \varepsilon_0} \frac{\partial B_z}{\partial x} \]

For \( x > 0 \), what component of field

\[ \frac{\partial B_z}{\partial t} - \frac{\partial E_y}{\partial x} = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 B_z}{\partial x^2} \quad \text{diffusion term} \]
\[ D = \frac{1}{\mu_0 \varepsilon_0} = \text{diffusion coefficient} \]

"Look for a solution of the form

\[ B_z(x, t) = B(t) \exp(-\mu x^2) \]
\[ B_z(x,t) = \int_0^x \frac{B_0}{\sqrt{\pi}} \exp\left(-\frac{x^2}{4t}\right) \, dx \]

How to find \( \frac{\partial B_z}{\partial x} \) and satisfy the D.E.

\[ u_0 T_y \sim \frac{\partial B_z}{\partial x} - \frac{A}{x} \exp\left(-\frac{x^2}{4Dt}\right) \]

Time Harmonic:

Suppose at \( t = 0 \), \( B(x=0,t) = B_0 \cos(\omega t) = \frac{1}{2} \Re\{B_0 e^{-i\omega t}\} \)

For \( x > 0 \), \( B_z = \frac{\omega}{k} \Re\{B(x)e^{-i\omega t}\} \)

Plug into diffusion:

\[ -i\omega B = D \frac{\partial^2 B}{\partial x^2} \]

\[ \hat{B}(k) = \hat{B}(0) \exp(-i\omega t) \]

\[ D k^2 B = -i\omega \]

\[ i\kappa = \sqrt{-\frac{\kappa}{D}} = \sqrt{-\frac{i\omega}{D}} = \frac{\sqrt{\omega}}{\sqrt{D}} \]

\[ 1 + \kappa^2 = \frac{1}{4} (1 + 2i + i^2) = 2i \]

\[ 1 - i \kappa = \frac{1 - i}{\sqrt{D} \omega} \]
\[ B_z(x,t) = \frac{B_0}{\delta} \text{Re}\left( B(0) \exp(-c \omega t + i - i) \delta \exp(-x/\delta) \right) \]

\[ \delta = \sqrt{\frac{c}{\omega}} \]

\[ \delta = \text{skin depth} \]

\[ B_z(x,t) = \frac{B_0}{\delta} \exp(-x/\delta) \cos(\omega t - x/\delta) \]
Surface Current Density

Surface Expander

\[ B(x,t) = \frac{\mu_0}{2} \text{Re} \{ \frac{1}{\mu_0} \hat{B}(x) \exp(-iKx) \} \]
\[ \hat{B}(x) = \hat{B}(0) \exp(-iKx) \]

\[ K = (1-\frac{c}{v}) \sqrt{\frac{\omega \mu_0}{2}} \quad d = \text{skin depth} = \frac{\sqrt{2}}{10} \mu_0 \]

Amper's Law
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \]
\[ J_y = \frac{1}{\mu_0} \frac{\partial}{\partial x} B_z \quad J_y(x,t) = \frac{1}{\mu_0} \text{Re} \left\{ \frac{iK}{\mu_0} \hat{B}(0) \exp(-i\omega t - iKx) \right\} \]

\[ E_y = \frac{J_y}{\sigma} \quad E_y(x,t) = \frac{1}{\mu_0} \text{Re} \left\{ \frac{iK}{\mu_0} \hat{B}(0) \exp(-i\omega t - iKx) \right\} \]

Surface Current Density

\[ K_y = \text{amps/mile} \quad B = H_\parallel \]

\[ K_y = \int_0^\infty dx J_y(x,t) = \int_0^\infty dx \left\{ -\frac{1}{\mu_0} \frac{\partial}{\partial x} B_z \right\} = B_z(0) \mu_0 = H_2(0) \]

\[ E_y(0,t) = \text{Re} \left\{ \frac{1}{\mu_0} \frac{iK}{\sigma} H_2(0) \right\} \quad 1K = \frac{1-i}{\sigma} \]

\[ E_y(0,t) = \frac{1-i}{\sigma} H_2(0) \]

\[ Z_s = \frac{1-i}{\sigma} = (1-i) \sqrt{\frac{\omega \mu_0}{20}} \]
**TABLE 8-1**

Skin Depths, $\delta$ (in mm), of Various Materials

<table>
<thead>
<tr>
<th>Material</th>
<th>$\sigma$ (S/m)</th>
<th>$f = 60$ (Hz)</th>
<th>$f = 1$ (MHz)</th>
<th>$f = 1$ (GHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>$6.17 \times 10^7$</td>
<td>8.27 (mm)</td>
<td>0.064 (mm)</td>
<td>0.0020 (mm)</td>
</tr>
<tr>
<td>Copper</td>
<td>$5.80 \times 10^7$</td>
<td>8.53</td>
<td>0.066</td>
<td>0.0021</td>
</tr>
<tr>
<td>Gold</td>
<td>$4.10 \times 10^7$</td>
<td>10.14</td>
<td>0.079</td>
<td>0.0025</td>
</tr>
<tr>
<td>Aluminum</td>
<td>$3.54 \times 10^7$</td>
<td>10.92</td>
<td>0.084</td>
<td>0.0027</td>
</tr>
<tr>
<td>Iron ($\mu \approx 10^3$)</td>
<td>$1.00 \times 10^7$</td>
<td>0.65</td>
<td>0.005</td>
<td>0.00016</td>
</tr>
<tr>
<td>Seawater</td>
<td>4</td>
<td>32 (m)</td>
<td>0.25 (m)</td>
<td>$^1$</td>
</tr>
</tbody>
</table>

$^1$ The $\epsilon$ of seawater is approximately $72 \epsilon_0$. At $f = 1$ (GHz), $\sigma/\omega \epsilon \approx 1$ (not $\approx 1$). Under these conditions, seawater is not a good conductor, and Eq. (8-57) is no longer applicable.

termine the attenuation constant, phase constant, intrinsic impedance, phase velocity, wavelength, and skin depth. (b) Find the distance at which the amplitude of $E$ is 1\% of its value at $z = 0$. (c) Write the expressions for $E(z, t)$ and $H(z, t)$ at $z = 0.8$ (m) as functions of $t$.

**Solution**

\[
\omega = 10^7 \pi \quad \text{(rad/s)},
\]

\[
f = \frac{\omega}{2\pi} = 5 \times 10^6 \quad \text{(Hz)},
\]

\[
\frac{\sigma}{\omega \epsilon \epsilon_0} = \frac{\sigma}{\omega \epsilon \epsilon_0 \epsilon_0} = \frac{4}{10^7 \pi \left(\frac{1}{36 \pi} \times 10^{-9}\right) 72} = 200 \gg 1.
\]

Hence we can use the formulas for good conductors.

**a) Attenuation constant:**

\[
\alpha = \sqrt{\frac{\mu \sigma}{\epsilon \sigma}} = \sqrt{\frac{5\pi 10^6 (4\pi 10^{-7}) 4}{8.89}} \quad \text{(Np/m)}.
\]

**Phase constant:**

\[
\beta = \sqrt{\frac{\mu \sigma}{\epsilon \sigma}} = 8.89 \quad \text{(rad/m)}.
\]

**Intrinsic impedance:**

\[
\eta_z = (1 + j) \sqrt{\frac{\mu \sigma}{\epsilon \sigma}}
\]

\[
= (1 + j) \sqrt{\frac{\pi (5 \times 10^6)(4\pi 10^{-7})}{4}} = \pi e^{\beta z} \quad \text{(\Omega)}.
\]

**Phase velocity:**

\[
u_p = \frac{\omega}{\beta} = \frac{10^7 \pi}{8.89} = 3.53 \times 10^6 \quad \text{(m/s)}.
\]