A non dispersive, nonlinear, isotropic, dielectric medium has the following constitutative relation,

\[ D = \epsilon_0 + \frac{\epsilon_0 \chi}{1 + \alpha|E|} E , \]

where \( \chi, \alpha \) are constants. By consideration of Poynting's theorem find expressions for the energy density in terms of the electric field for the two cases, \( \rho = 1 \) and \( 2 \). Show that they reduce to the standard expression when the electric field is small. If one thinks of the polarization as being determined by balancing the stretching force of an electric field against the restoring force of a molecular spring, what is the qualitative relation between \( \alpha \) and the spring constant?

2) 1. An electromagnetic wave with electric field,

\[ \textbf{E} = \text{Re} \{ \hat{E}_y(x,z) \hat{\textbf{y}} \exp(-i\omega t) \} , \]

propagates through a medium with dielectric constant \( \epsilon(x) \) and permeability \( \mu_0 \).

A) Derive from Maxwell's Equations, a differential equation for the complex amplitude \( \hat{E}_y(x,z) \). Assume that there is no variation of any quantity with \( y \), and the dielectric constant depends only on \( x \).

B) Obtain expressions for the components of the magnetic field associated with this wave. Verify that all of Maxwell's Equations are satisfied.

C) Show that solutions, which are nearly plane waves, (viz. \( \hat{E}_y(x,z) = \tilde{E}_y(x,z) \exp(ik_0z) \) where \( k_0 >> \partial / \partial z \)) satisfy a Shrödinger-like equation,
\[
2ik_0 \frac{\partial}{\partial z} \hat{E}_y(x,z) + \frac{\partial^2}{\partial x^2} \hat{E}_y(x,z) - V(x)\hat{E}_y(x,z) = 0
\]

What is the potential \(V(x)\) in terms of the spatially varying dielectric constant and other quantities?

D) If the medium is homogeneous (\(\varepsilon\) is a constant) a Guassian shaped wave spreads in \(x\) as it propagates in \(z\). This is known as diffraction.

\[
\hat{E}_y(x,z) = \frac{E_0}{[\pi\sigma(z)]^{1/2}} \exp[-x^2 / \sigma(z)]
\]

Obtain an expression for the \(z\) dependent complex width, \(\sigma(z)\) in terms of its initial value \(\sigma(0)\), and the wave number \(k_0\). (Hint: Substitute the Gaussian expression above into the Shrödinger equation and pick \(k_0\) such that \(V = 0\). and match powers of \(x\).)

E) How must the dielectric constant depend on \(x\), and what must the wave number \(k_0\) be in order that the wave does not spread as it propagates \(\sigma(z) = \sigma(0)\)? Interpret your result.