10.1 In-Plane Field of a Current Strip

This problem amounts to superposing the fields from a collection of long, straight wires. The surface current density is \( \mathbf{K} = (I/b)\hat{z} \). Therefore, an infinitely long filament at \( y \) with width \( dy \) carries a current \( dI = K dy \). Treating this as a wire gives a contribution to the magnetic field of

\[
d\mathbf{B} = -\frac{\mu_0 K dy}{2\pi (a + b - y)} \hat{x}.
\]

Therefore, the total field at the observation point is

\[
\mathbf{B} = -\frac{\mu_0 I}{2\pi b} \int_0^b \frac{dy}{a + b - y} \hat{x} = -\frac{\mu_0 I}{2\pi b} \ln \left( \frac{a + b}{a} \right) \hat{x}.
\]

10.22 The Magnetic Field of Charge in Uniform Motion

(a) The relevant current density is \( \mathbf{j}(\mathbf{r}) = \mathbf{v} \rho(\mathbf{r}) \) so

\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{\mu_0 \mathbf{v}}{4\pi} \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{\mathbf{v}}{c^2} \varphi(\mathbf{r})
\]

\[
\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{c^2} \nabla \times (\mathbf{v} \varphi) = -\frac{1}{c^2} \mathbf{v} \times \nabla \varphi = \frac{1}{c^2} \mathbf{v} \times \mathbf{E}.
\]

(b) From Gauss’ law, the electric field of a line charge (charge/length \( \lambda \)) coincident with the \( z \)-axis is

\[
\mathbf{E}(\rho) = \frac{\lambda}{2\pi \epsilon_0 \rho} \hat{\rho}.
\]
We get a flowing current \( I = \rho v \) if \( \mathbf{v} = v \hat{z} \), so, from (a), the magnetic field is

\[
\mathbf{B}(\rho) = \frac{\mu_0 \lambda v}{2\pi \rho} \hat{z} \times \hat{\rho} = \frac{\mu_0 I}{2\pi \rho} \hat{\phi}.
\]

This agrees with Ampère's law. Similarly, Gauss' law gives the electric field of a sheet with uniform charge per area \( \sigma \) coincident with \( x = 0 \) as

\[
\mathbf{E}(x) = \hat{x} \frac{\sigma}{2\epsilon_0} \text{sgn}(x).
\]

If the surface current density is \( \mathbf{K} = \sigma \mathbf{v} \) where \( \mathbf{v} = v \hat{z} \), part (a) gives the Ampère's law result

\[
\mathbf{B}(x) = y \frac{K \mu_0}{2} \text{sgn}(x).
\]

10.25 Toroidal and Poloidal Magnetic Fields

(a) Use the fact that \( \nabla \cdot \nabla \times \mathbf{Q} = 0 \) for any vector \( \mathbf{Q} \). This gives \( \nabla \cdot \mathbf{P} = \nabla \cdot \nabla \times \mathbf{L} \gamma = 0 \) immediately. Similarly, \( \mathbf{T} = \mathbf{L} \psi = -i\mathbf{r} \times \nabla \psi = i\nabla \times (\psi \mathbf{r}) \) so \( \nabla \cdot \mathbf{T} = 0 \) as well.

(b) Suppose \( \mathbf{B} \) is toroidal so \( \mathbf{B} = \mathbf{L} \psi \). This implies that \( \mathbf{j} \) is poloidal because \( \mu_0 \mathbf{j} = \nabla \times \mathbf{B} = \mu_0 \nabla \times \mathbf{L} \xi \). Conversely, suppose \( \mathbf{B} \) is poloidal so \( \mathbf{B} = \nabla \times \mathbf{L} \psi \). In that case, \( \mathbf{j} \) is toroidal because

\[
\mu_0 \mathbf{j} = \nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{L} \psi) = \nabla (\nabla \cdot \mathbf{L} \psi) - \nabla^2 \mathbf{L} \psi = -\nabla^2 \mathbf{L} \psi = -\mathbf{L} \nabla^2 \psi.
\]

(c) The magnetic field of a toroidal solenoid was found in the text to be

\[
\mathbf{B}(\rho, z) = \begin{cases} 
\frac{\mu_0 NI}{2\pi \rho} \hat{\phi} & \text{for points inside the torus,} \\
0 & \text{for points not outside the torus.}
\end{cases}
\]

Therefore, if \( C \) is a constant, we need to show that a function \( \psi(\mathbf{r}) \) exists such that

\[
\mathbf{B} = \frac{C}{\rho} \hat{\phi} = \mathbf{L} \psi = i\nabla \psi \times \mathbf{r}.
\]
Now, \( \mathbf{\hat{r}} \times \mathbf{\hat{\theta}} = \mathbf{\hat{\phi}} \). Comparing this with the equation above and switching from cylindrical to polar coordinates tells us that

\[
-i \frac{\partial \psi}{\partial \theta} = \frac{C}{\rho} = \frac{C}{r \sin \theta}.
\]

Integration gives \( \psi(r, \theta) = \frac{i}{r} \ln \tan \frac{\theta}{2} \), which proves the assertion.

(d) We have \( \nabla \cdot \mathbf{B} = 0 \) and \( \nabla \times \mathbf{B} = 0 \) in \( V \). The Helmholtz theorem would give \( \mathbf{B} \equiv 0 \) if \( V \) were all of space. When \( V \) is finite, the double-curl identity tells us that

\[
0 = \nabla \times (\nabla \times \mathbf{B}) - \nabla (\nabla \cdot \mathbf{B}) = -\nabla^2 \mathbf{B}.
\]

Therefore, \( \nabla^2 \mathbf{B} = 0 \) in \( V \).

(e) We have

\[
\mathbf{B} = \nabla \times \mathbf{A}
\]

\[
= \nabla \times \mathbf{L} \psi + \nabla \times (\nabla \times \mathbf{L} \gamma)
\]

\[
= \nabla \times \mathbf{L} \psi + \nabla (\nabla \cdot \mathbf{L} \gamma) - \nabla^2 \mathbf{L} \gamma
\]

\[
= \nabla \times \mathbf{L} \psi - \nabla^2 \mathbf{L} \gamma.
\]

Now take the Laplacian of both sides. We get \( \nabla^2 \mathbf{B} = 0 \) if \( \psi(x) \) and \( \gamma(x) \) both satisfy Laplace’s equation, i.e., \( \nabla^2 \psi = 0 \) and \( \nabla^2 \gamma = 0 \). In that case, \( \nabla^2 \mathbf{L} \gamma = \mathbf{L} \nabla^2 \gamma = 0 \) so we are left with \( \mathbf{B} = \nabla \times \mathbf{L} \psi \), which implies that the vector potential \( \mathbf{A} = \mathbf{L} \psi \) in \( V \).

11.6 A Spinning Spherical Shell of Charge

(a) The surface current density is \( \mathbf{K} = \sigma \mathbf{v} \) where \( \sigma = Q/4\pi R^2 \) is the surface charge density and \( \mathbf{v} = \mathbf{\omega} \times \mathbf{r} \) is the velocity of a point on the surface at the point \( \mathbf{r} \). The corresponding volume current density requires a delta function to define the surface:

\[
\mathbf{j} = \frac{Q}{4\pi R^2} (\mathbf{\omega} \times \mathbf{r}) \delta(r - R).
\]

On the other hand,

\[
\mathbf{j} = \nabla \times [\mathbf{M} \Theta (R - r)] = \mathbf{M} \times \nabla \Theta (r - R) = \mathbf{M} \times \mathbf{\hat{r}} \delta(r - R) = \mathbf{M} \times \frac{\mathbf{r}}{R} \delta(r - R).
\]

Comparing these two formula shows that \( \mathbf{M} = (Q/4\pi R) \mathbf{\omega} \).
(b) Using one of the expressions for $\mathbf{j}$ just above and the definition of magnetic moment,

$$
\mathbf{m} = \frac{1}{2} \int d^3 r \mathbf{r} \times \mathbf{j} = \frac{1}{2} \int d^3 r \mathbf{r} \times (\mathbf{M} \times \hat{\mathbf{r}}) \delta(r - R)
$$

$$
= \frac{1}{2} \int d^3 r \left[ \mathbf{M} (\mathbf{r} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}} (\mathbf{M} \cdot \mathbf{r}) \right] \delta(r - R).
$$

To do the second part of the last integral, choose $\mathbf{M} = M \hat{\mathbf{z}}$ and note that only the $\cos \theta \hat{\mathbf{z}}$ part of $\hat{\mathbf{r}}$ survives the integration over $\phi$. Hence,

$$
\mathbf{m} = \frac{1}{2} \int d^3 r \mathbf{M} r \delta(r - R) - \frac{1}{2} \hat{\mathbf{z}} M \int_0^\infty drr^2 \int_0^{2\pi} d\phi \int_{-1}^1 d(\cos \theta) \cos^2 \theta M r \delta(r - R)
$$

$$
= \frac{1}{2} \left[ 4\pi R^3 M - 2\pi M R^3 \times \frac{2}{3} \right] \hat{\mathbf{z}}
$$

$$
= \frac{4}{3} \pi R^3 M = \frac{1}{3} QR^2 \omega.
$$

(c) To get the vector potential, we integrate by parts:

$$
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{j}(\mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|} = \mathbf{M} \times \frac{\mu_0}{4\pi} \int d^3 r' \frac{\nabla' \Theta(\mathbf{r'} - R)}{|\mathbf{r} - \mathbf{r'}|} = \mu_0 \mathbf{M} \times \frac{1}{4\pi} \int_{r' < R} d^3 r' \frac{\mathbf{r} - \mathbf{r'}}{|\mathbf{r} - \mathbf{r'}|}.
$$

The integral is the electric field of a ball of radius $R$ with uniform charge density $\rho(\mathbf{r'}) = \epsilon_0$. This we compute straightforwardly using Gauss’ law. Therefore,

$$
\mathbf{A}(\mathbf{r}) = \begin{cases} 
\frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} & r > R, \\
\frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{R^3} & r < R.
\end{cases}
$$

(d) Outside the sphere, we get a pure dipole field with magnetic dipole moment $\mathbf{m}$. Inside the sphere, we get a uniform field:

$$
\mathbf{B}(r < R) = \frac{\mu_0}{4\pi R^3} \nabla \times (\mathbf{m} \times \mathbf{r}) = \frac{\mu_0}{2\pi R^3} \mathbf{m} = \frac{\mu_0}{6\pi R^3} \omega.
$$
11.11 Purcell’s Loop

(a) The figure below shows that Purcell’s loop is equivalent to the sum of three square loops of current. This is so because no magnetic field is produced by the two pairs of oppositely directed and spatially coincident lines of current. Each square loop carries a magnetic dipole moment with magnitude \( I(2b)^2 \). The moments contributed by the top and bottom loops are oppositely directed and thus cancel. Therefore, using the right-hand rule, the magnetic dipole moment of Purcell’s loop is \( \mathbf{m} = 4Ib^2 \hat{y} \).

![Diagram of Purcell’s Loop](image)

(b) By analogy with the electric case, the top and bottom square loops cancel for the dipole moment but add for the quadrupole moment. Thus, we expect a non-negligible magnetic quadrupole moment for the Purcell loop.

MATLAB PROBLEM

![MATLAB Graph](image)