ENEE 680 Homework 1 Solution

Zangwill 4.3/ Vector identities.

(a) General approach is to use (1.29) for the dot product and (1.37) for the cross product.

\[ (1.29) \quad \vec{U} \cdot \vec{F} = V_k F_k \]
\[ (1.37) \quad (\vec{V} \times \vec{F})_i = \epsilon_{ijk} V_j F_k. \]

so \[ (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \times \vec{B})_i (\vec{C} \times \vec{D})_i \]
\[ = \epsilon_{ijk} A_j B_k \epsilon_{imn} C_m D_n. \]

Now use (1.39) \[ \epsilon_{ijk} C_{ist} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks} \]
\[ \Rightarrow (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) A_j B_k C_m D_n \]
\[ = A_j B_k C_j D_k - A_j B_k C_k D_j \]
\[ = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \]
(b) $\nabla \cdot (\vec{f} \times \vec{g}) = \nabla \cdot (\vec{f} \times \vec{g}) = \nabla \cdot \epsilon_{ijk} f_j g_k$

change order $ijk \rightarrow kij \rightarrow (-) jik$

$= g_k (\nabla \times \vec{f})_k - f_j (\nabla \times \vec{g})_j$

$= \vec{g} \cdot (\nabla \times \vec{f}) - \vec{f} \cdot (\nabla \times \vec{g})$

Zangwill 1.04 | Use (1.30) $\nabla \varphi = \hat{e}_k \partial_k \varphi$

(a) $\nabla \cdot (\vec{f} \vec{g}) = \partial_i (f g_i) = f \partial_i g_i + g_i \partial_i f$

$= f \nabla \cdot \vec{g} + \vec{g} \cdot \nabla f$

Note that $g_i \partial_i f$ is more like $(\vec{g} \cdot \nabla) f$ since there is no directional vector as in (1.30).

(b) $[\nabla \times (\vec{f} \vec{g})]_i = \epsilon_{ijk} \nabla \cdot (\vec{f} g_k)$

$= f \epsilon_{ijk} \partial_j g_k + \epsilon_{ijk} (\partial_i f) g_k$

$= f (\nabla \times \vec{g})_i + (\nabla f \times \vec{g})_i$

$\Rightarrow \nabla \times (\vec{f} \vec{g}) = f (\nabla \times \vec{g}) + \nabla f \times \vec{g}$
Zangwill 3.4 | Charges in a Line.

Equation (3.8):

\[ \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{d^3 \mathbf{r} \cdot \rho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|^3} \]

To simplify it, let the charge sit at \( \mathbf{r}' = 0 \)

\[ \Rightarrow \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{d^3 \mathbf{r} \cdot \rho(\mathbf{r})}{|\mathbf{r}|^3} \delta(\mathbf{r}) \]

\[ = \frac{q}{4\pi \varepsilon_0} \hat{\mathbf{r}} \]

Clearly, \( \mathbf{E} \) has the direction of \( \mathbf{r} \) if \( q > 0 \) and opposite of \( \mathbf{r} \) if \( q < 0 \).

So:

\[ \begin{align*}
&\begin{array}{c}
\text{observer at } \mathbf{r}
\end{array} \\
&\begin{array}{c}
\text{observer at } \mathbf{F}
\end{array}
\end{align*} \]

Now, 5 charges on the line with equal spacing:
Zangwill 3.5f  Gauss' Law

Eqn (3.39): \( \int_S d\vec{S} \cdot \vec{E}(\vec{r}) = Q_v \)

Symmetry of a specific problem is used to simplify the dot product \( d\vec{S} \cdot \vec{E} \) to a constant. This is usually done through analyzing the dependence of \( \vec{E} \) on spatial variation.

\[(a) \quad f(x) = f_0 e^{-k|x|} \]

Since \( f(x) \) is invariant with respect to changes in \( y \) and \( z \), we can make our Gaussian box in Cartesian coordinate with a fixed surface area perpendicular to \( x \) and vary its length to observe the change in \( f(x) \). To calculate \( Q_v \)

\[
\overline{S}_x = \pm A \ \hat{x}
\]

Since \( \overline{E}(x) = E(x) \hat{\kappa} \) and only passes through surface \( A \) as showed

\[
\int_{\overline{S}} d\overline{S} \cdot \overline{E} = 2AE
\]

Now, the charge enclosed is \( Q_v = \int A f(x) dx \)
Since the charge distribution varies according to \( x \)-position, the electric field at a certain position \( x \) is determined by charge distribution up to that point. Let the variation be \( x', \,-x \leq x' \leq x \). For \( x > 0 \):

\[
Q_v = A \Phi_0 \int_{-x}^{x} dx' e^{-k |x'|}\\
= 2A \Phi_0 \int_{0}^{x} dx' e^{-k x'} = 2A \Phi_0 \left(1 - e^{-k x}\right)\\
\]

For \( x < 0 \):

\[
Q_v = A \Phi_0 \int_{-x}^{x} dx' e^{-k |x'|}\\
= 2A \Phi_0 \int_{0}^{x} dx' e^{k x'} = \frac{2A \Phi_0}{k} (e^{k x} - 1)\\
= \frac{2A \Phi_0}{k} (1 - e^{k x})\\
\]

\[
E(x > 0) = \frac{2 \Phi_0}{k e_0} (1 - e^{-k x})\\
E(x < 0) = -\frac{2 \Phi_0}{k e_0} (1 - e^{k x})
\]
(b) \( f(x, y) = f_0 e^{-k \sqrt{x^2 + y^2}} \)

Let \( r^2 = x^2 + y^2 \)

\[ f(r) = f_0 e^{-kr} \]

In cylindrical coordinates, \( \bar{E} = E(r) \bar{r} \) due to the symmetry of the problem.

\[ \Rightarrow \text{Make a cylindrical Gaussian surface with fixed surface area perpendicular to length } L. \]

\[ \int \bar{E} \cdot d\bar{S} = 2\pi r L E(x) \]

Similar to part (a):

\[ Q = 2\pi f_0 L \int_0^r r' e^{-kr'} \, dr' \]

Integrate by parts:

\[ \int_0^r r' e^{-kr'} \, dr' = \frac{r' e^{-kr'}}{-k} \bigg|_0^r - \int_0^r \frac{e^{-kr'}}{-k} \, dr' \]

\[ = \frac{r e^{-kr}}{-k} - \frac{e^{-kr}}{k^2} \bigg|_0^r = -\frac{e^{-kr}}{k} \left( \frac{1}{k} + r \right) + 1 \]
\[ Q_0 = \frac{2\pi \rho_0 L}{k^2} \left( 1 - e^{-kr} - kre^{-kr} \right) \]

\[ \bar{E}(r) = \frac{\rho_0}{\epsilon_0 k^2 r} \left( 1 - e^{-kr} - kre^{-kr} \right) \]

(c) \( \rho(r) = \rho_0 e^{-kr} \) in spherical coordinate

Similar to the previous parts, \( \bar{E} = E(r) \hat{r} \), and we can pick the Gaussian surface around the charge.

\[ \int_{S} d\vec{S} \cdot \bar{E} = 4\pi r^2 E(r) \]

\[ Q_0 = \frac{4\pi \rho_0}{4} \int_{0}^{r} dr' r'^2 e^{-kr'} \]

The integral can be done by parts twice to give

\[ Q_0 = \frac{4\pi \rho_0}{k^3} \left[ 1 - e^{-kr} \left( 1 + kr + \frac{1}{2} \frac{k^2 r^2}{2} \right) \right] \]

\[ \Rightarrow \bar{E}(r) = \frac{\rho_0}{\epsilon_0 k^2 r} \left[ 1 - e^{-kr} \left( 1 + kr + \frac{1}{2} \frac{k^2 r^2}{2} \right) \right] \]
Zangwill 3.11/ Charged disk.

(a) Potential on symmetry axis

\[ \phi(z) = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} d\psi \int_0^R dr \frac{r}{\sqrt{r^2+z^2}} \]

\[ = \frac{\sigma}{4\pi\varepsilon_0} \frac{2\pi}{2} \int_0^R dr \frac{r}{\sqrt{r^2+z^2}} \]

The radial integral can be done by changing \( u = r^2 + z^2 \)

\[ \Rightarrow \frac{du}{dr} = 2r \Rightarrow r\,dr = \frac{du}{2} \]

\[ \Rightarrow \phi(z) = \frac{\sigma}{2\varepsilon_0} \left[ \sqrt{R^2+z^2} - \sqrt{z^2} \right] \]
(b) Following the hint:

\[ \psi(A) = \frac{1}{2\pi \varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_{0}^{2R \cos \psi} r dr \]

Use a sweeping vector \( \mathbf{r} \) to scan all the disk area.

\[ \psi = \frac{1}{2\pi \varepsilon_0} \int_{0}^{2R \cos \psi} r dr \]

Upper \( r \) is from \( ds = r dr d\theta \)

Lower \( r \) is from \( \psi = \frac{5\psi}{2} \)

(c) Since \( \psi > 0 \), \( \psi(0) = \frac{5R}{2\varepsilon_0} \) from (a)

\[ \psi(A \text{ on rim}) = \frac{5R}{2\varepsilon_0} \] from (b)

\( \psi \) is uniform on rim.

\( \Rightarrow \psi \) decreases as \( r \) increases.

\[ \mathbf{E} = -\nabla \psi \Rightarrow \mathbf{E} \text{ points outward.} \]

\[ \text{with equal strength in every direction.} \]
(a) Calculate total energy of disk

\[ U_E = \frac{1}{2} \int d^3r \, \phi(r) \cdot \phi(r) \]

The electrostatic energy is defined as the amount of work required to assemble a charge distribution from infinity. So \( U_E = \) work to bring all the infinitesimally small pieces of charge to form the ring.

Naturally, the disk grows bigger and bigger from \( r = 0 \) to \( r = R \). Note that the work at the very beginning is close to zero since there is no resistance force. So, double counting doesn't work.

From (b), \( \phi(r) = \frac{\varphi r}{\pi \varepsilon_0} \).

The charge at a ring's edge is \( dq = 0^2 \pi r dr \).

\[ U_E = \int_0^R dr \left( \frac{0^2 \pi r}{\pi \varepsilon_0} \right) \frac{\varphi r}{\pi \varepsilon_0} = \frac{\varphi^2 R^3}{3 \varepsilon_0} \]
Zangwill 3.13 | Uniform charge cube.

Following the hint, assume the big cube can be formed by 8 small cubes.

For an arbitrary cube with size \( S \); the potential at the corner is:

\[
\Phi_{\text{corner}} = \frac{1}{4\pi\varepsilon_0} \int_0^S \int_0^S \int_0^S \, dx \, dy \, dz \frac{1}{\sqrt{x^2 + y^2 + z^2}}.
\]

For a small cube with size \( S/2 \):

\[
\Phi_{\text{corner}} = \frac{S}{4\pi\varepsilon_0} \int_0^{S/2} \int_0^{S/2} \int_0^{S/2} \, dx \, dy \, dz \frac{1}{\sqrt{x^2 + y^2 + z^2}}.
\]

\[
= \frac{S}{4\pi\varepsilon_0} \frac{1}{8} \int_0^S \int_0^S \int_0^S \, dx \, dy \, dz \frac{1}{\sqrt{x^2 + y^2 + z^2}}.
\]

\[
= \frac{1}{4} \Phi_{\text{corner}} = \frac{1}{4} \Phi_1
\]

Now, if we put 8 small cubes together, then

\[
\Phi_0 = 8 \times \Phi_{\text{small}} = 2 \Phi_1 \Rightarrow \frac{\Phi_0}{\Phi_1} = 2
\]
Zangwill 3.15 / Electrostatic Energy

(a) Total energy $U_E$ of the hollow sphere.

Eqn gives

$$U_E = \frac{1}{2} \frac{1}{2} \int d^3 r \left| \mathbf{E} \right|$$

We want to work out the electric field because the charge and spherical symmetry facilitate easy calculation. Using charge distribution and potential is much harder with a hollow sphere.

To achieve the total energy, we are interested in the electric field in all regions, as the energy is defined as amount of work to assemble the hollow sphere. Due to symmetry, the electric fields inside the hollow $(r \leq a)$ cancel out, $E(r \leq a) = 0$.

In the region $r > R$, $E = \frac{Q}{4 \pi \epsilon_0 r^2}$ from Gauss' law.
or more specifically eqn (3.40).

The same equation gives $E(\alpha < r < R)$ as

$$E = \frac{Q_e(r)}{4\pi \epsilon_0 r^2}$$

where $Q_e(r)$ is the charge enclosed by the Gauss' surface.

The volume that is enclosed which contains charge is

$$V_{en} = \frac{4\pi}{3} (r^3 - \alpha^3)$$

The total volume of charge is

$$V_T = \frac{4\pi}{3} (R^3 - \alpha^3)$$

Since the charge is distributed uniformly,

$$\frac{Q_e(r)}{Q} = \frac{r^3 - \alpha^3}{R^3 - \alpha^3}$$

So, the total energy is:

\[
U_E = \frac{\epsilon_0}{2} \int d^3r |E| = \frac{\epsilon_0}{2} \frac{1}{(4\pi \epsilon_0)^2} \int \frac{Q^2}{\text{charge distribution}}
\]

\[
= \frac{Q^2}{4\pi \epsilon_0} \frac{1}{2} \left\{ \int_\alpha^R \frac{r^3 - \alpha^3}{(R^3 - \alpha^3)^2} \frac{r^2}{r^4} + \int_\alpha^R \frac{r^2}{r^4} \right\}
\]
\[ U_E = \frac{Q^2}{4\pi \varepsilon_0} \frac{1}{2} \left\{ \frac{1}{(R^3 - a^3)^2} \int_a^R \frac{dr}{r^2} + \int_R^\infty \frac{dr}{r^2} \right\} \]

Let \( x = \frac{a}{R} \). Mathematica algebra gives

\[ U_E = \frac{1}{4\pi \varepsilon_0} \frac{Q^2}{2R} \left\{ 1 + \frac{1 - 5x^6 + 9x^4 - 3x^2}{5(1-x^3)^2} \right\} \]

\( a = 0, \ x = 0 \) gives \( U_E = \frac{1}{4\pi \varepsilon_0} \frac{3Q^2}{5R} \),

\( \Rightarrow \) uniform charge sphere solution, eqn (3.77)

\( a = R, \ x = 1 \), \( \lim_{x \to 1} U_E = \frac{1}{4\pi \varepsilon_0} \frac{Q^2}{2R} \)

\( \Rightarrow \) hollow shell solution

Check: \( E(x) = \frac{Q}{4\pi \varepsilon_0 r^2} \rightarrow U_E = \frac{1}{4\pi \varepsilon_0} \frac{1}{2} \int_0^R \int_1^\infty \frac{|E|^2}{r^2} d\Omega \), \( R < r < \infty \)

\[ = \frac{1}{4\pi \varepsilon_0} \frac{Q^2}{2R} \]
Zangwill 3.16 / Interaction Energy of Spheres

(a) \[ U_E = \frac{1}{2} \int d^3r \rho(r) \Phi(r) \]

\[ \Rightarrow V_E = \frac{1}{2} \int d^3r \rho_1(r) \Phi_2(r) + \frac{1}{2} \int d^3r \rho_2(r) \Phi_1(r) \]

charge density on 1 potential from 2 to 1

Due to symmetry, \[ V_E = \int d^3r \rho_1(r) \Phi_2(r) \]

\[ = \int_{S_1} ds \rho_1(s) \Phi_2(s) \]

Since the charge is only distributed only on the sphere's surface, \[ S_1 = \frac{Q}{4\pi R^2} \text{ independent of } r \]

\[ \Phi_2 = \frac{1}{4\pi \varepsilon_0} \int \frac{S_{22}(r)}{|r-r'|} dr = \frac{a}{4\pi \varepsilon_0} \int \frac{dr}{|r-r'|} \]

Using symmetry of the problem, we can calculate \( V_E \) for 1 infinitesimal point on sphere 1 then do another integral to
collect the total $V_E$.

then, total $V_E = \frac{Q^2}{(4\pi)^2 R^2 \varepsilon_0} \int_0^{2\pi} R^2 d\phi \int_{\phi}^{\frac{\pi}{2}} d(\cos \theta) \frac{1}{r - r'}$

$\vert r - r' \vert = \sqrt{R^2 + d^2 - 2Rd \cos \theta}$ by triangular trigonometry

$\Rightarrow V_E = \frac{Q^2}{(4\pi)^2 \varepsilon_0} \int_{-1}^{1} \frac{dx}{\sqrt{R^2 + d^2 - 2Rxdx}} = \frac{Q^2}{4\pi \varepsilon_0 d}$

(b) The interaction energy is independent of the spheres' radius because they have effect of point charges.

\[ S_S = \frac{Q}{4\pi R^2} \]

From the schematic above, we can say that the symmetry induces the total charge effect instead of the charge-distribution on the surface. This is just similar to Earth's gravitational field.

Hydrogen atom \( \rightarrow \) point nucleus with charge \( +1 e \)

electron charge distribution: \( f_-(r) \)

\[
f_-(r) = - \frac{1e}{\pi a^2 r} e^{-2r/a}
\]

Typically, the energy of the system consists of

1. The self-energy of the electron(s)
2. The self-energy of the nucleus
3. The interaction energy between them.

Since the energy of the system is defined as the work required to assemble it, the work that we need to do to bring the electron away is negative of the total energy except for the self-energy of the nucleus (it has been there and will be there).
So,

\[ I = -\frac{1}{4\pi\varepsilon_0} \int d^3r \int d^3r' \frac{\rho_+(r')\rho_-(r)}{|r-r'|} - \frac{1}{2} \int d^3r \rho_+(r) \rho_-(r) \]

\[ \rho_+(r) = +|e| \delta(r) \]

So, the interaction energy part is:

\[ I_{\text{int}} = \frac{|e|}{4\pi\varepsilon_0} \int d^3r \frac{|e|}{\pi \alpha^2 r^2} e^{-\frac{2r}{\alpha}} = \frac{e^2}{2\pi\varepsilon_0 \alpha} \]

The self-energy of the electron is:

\[ I_e = -\frac{1}{2} \int d^3r \rho_-(r) \rho_-(r) \]

with \( \rho_-(r) = \int_0^\infty dr' E(r') \), from \( E = -\nabla \Phi \)

\[ \Phi = \int_0^r \frac{4\pi \varepsilon_0}{4\pi \varepsilon_0} \int_0^r d\rho' d\phi' \]

- Find \( E(r) \) using Gauss's law:

\[ E = \frac{Q(r)}{4\pi \varepsilon_0 r^2} \text{ for a charge sphere} \]

- we can use the same approach since electron it can be
Problem A / In Spherical Coordinate

In plasma, the potential surrounding a point charge \( q \) is:

\[
\phi_0 (\vec{x}) = \frac{q}{\epsilon_0} \frac{e^{-r/2d}}{r} \frac{1}{\ln r_0 / \ln r}
\]

Simplified:

\[
\phi_0 (r) = \frac{q}{\epsilon_0} \frac{e^{-r/2d}}{r}, \text{ due to symmetry.}
\]

So I care about distance, not direction.

Poisson's equation:

\[
\varphi (r) = -\epsilon_0 \nabla^2 \phi
\]

In spherical coordinate:

\[
\nabla^2 \Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right)
\]

\[
\frac{\partial \phi}{\partial r} = -\frac{q}{\epsilon_0} \frac{e^{-r/2d}}{r} \left( \frac{1}{r} + \frac{1}{1d} \right)
\]

\[
\frac{r^2 \partial \phi}{\partial r} = -\frac{q}{\epsilon_0} \frac{e^{-r/2d}}{1d} \left( 1 + \frac{r}{1d} \right)
\]

\[
\frac{2}{\partial r} \frac{r^2 \partial \phi}{\partial r} = -\frac{q}{\epsilon_0} \frac{e^{-r/2d}}{1d} \left\{ \left( \frac{1 + r}{1d} \right) \left( -\frac{1}{2d} \right) + \frac{1}{1d} \right\}
\]

\[
\Rightarrow \varphi (r) = -\epsilon_0 \frac{4}{\partial r} \frac{r^2 \partial \phi}{\partial r} = -\frac{q}{\epsilon_0} \frac{e^{-r/2d}}{1d} \frac{1}{2d} \left| \phi_0 \right|
\]
Problem A: In Cartesian.

In plasma, the potential surrounding a point charge $q$ is

$$
\phi_0(x) = \frac{q}{4\pi \varepsilon_0} \frac{-|\vec{x} - \vec{x}_0|}{|\vec{x} - \vec{x}_0|}
$$

Simplified:

$$
\phi_0(r) = \frac{q}{4\pi \varepsilon_0} \frac{e^{-r/l_d}}{r}
$$

due to symmetry.

Poisson's equation:

$$
\nabla^2 \phi = -\varepsilon_0 \nabla^2 \phi
$$

$$
\nabla^2 \Rightarrow \frac{d^2}{dr^2}
$$

$$
\frac{d\phi}{dr} = \frac{-q}{4\pi \varepsilon_0} \left( \frac{1 + \frac{1}{r^2}}{r^2} \right)^{r/l_d}
$$

$$
\frac{d^2\phi}{dr^2} = \frac{-q}{4\pi \varepsilon_0} 
\left\{ \left( \frac{1 + \frac{1}{r^2}}{r^2} \right) \frac{1}{r} + \frac{2 + \frac{1}{r^2}}{r^2} \right\}^{r/l_d}
$$

$$
\Rightarrow \phi(r) = \frac{-q}{4\pi \varepsilon_0} \frac{e^{-r/l_d}}{r} \left\{ \frac{2}{r^2} + \frac{1}{r^2} + \frac{2}{r^2} \right\}
$$

Examine

$$
\frac{2 + \frac{2}{r^2} + \frac{1}{r^2}}{r^2} = \frac{2l_d + 2r_{ld} + r^2}{r^2 l_d}
$$
\[
- \frac{2 \left( \frac{2d}{r} \right)^2 + 2 \frac{2d}{r} + 1}{1d^2}
\]

If \( r \gg 1d \) then \( \frac{2d}{r} \to 0 \)

\[
\phi(r) \sim -\frac{q}{4\pi \varepsilon_0} e^{-\frac{r(2d)}{4\pi \varepsilon_0}} = -\frac{\phi_0}{1d}
\]

\( \to \) charge distribution is proportional to potential.
Prob. A) \[ x = d \]
\[ x = 0 \]
Volume Charge density: \[ P(x) = \frac{P_0 + P_1 x}{\varepsilon_0} \]

Poisson's Equ: \[ \nabla^2 \phi = -\frac{P}{\varepsilon_0} \]

where \( \phi = \) Potential

This problem is 1D so \[ \nabla^2 \phi = \frac{d^2 \phi(x)}{dx^2} \]

\[ \rightarrow \frac{d^2 \phi(x)}{dx^2} = -\frac{P(x)}{\varepsilon_0} = -\left[ \frac{P_0 + P_1 x}{\varepsilon_0} \right] \]

W/ Boundary:
\[ \phi(x=0) = \phi_{\text{in}} \]

Conditions:
\[ \phi(x=d) = \phi_{\text{out}} \]

Integrating both sides of Equ. 1 w/ respect to \( x \),

\[ \frac{d\phi(x)}{dx} = -\left[ \frac{P_0 x + P_1 x^2}{2 \varepsilon_0} \right] + C \]

where \( C = \) integration constant

Integrating both sides of Equ. 2 w/ respect to \( x \),

\[ \phi(x) = -\left[ \frac{P_0 x^2 + P_1 x^3}{6 \varepsilon_0} \right] + C x + C_1 \]

where \( C_1 = \) another integration constant

Apply boundary conditions to Equ. 3 to determine \( C \) and \( C_1 \):

\[ \phi(x=0) = 0 = C_1 \rightarrow C_1 = 0 \]

\[ \phi(x=d) = V = -\frac{P_0 d^2}{2 \varepsilon_0} - \frac{P_1 d^3}{6 \varepsilon_0} + C d \]

\[ \rightarrow C = \frac{V}{d} + \frac{P_0 d}{2 \varepsilon_0} + \frac{P_1 d}{6 \varepsilon_0} \]
Thus potential is:

\[
\phi(x) = -\frac{p x^3}{6 \varepsilon_0 d} - \frac{p_0 x^2}{2 \varepsilon_0} + \left[ \frac{V}{d} + \frac{p_0 d}{2 \varepsilon_0} + \frac{p_1 d}{6 \varepsilon_0} \right] x
\]