1a) \( \nabla \cdot \mathbf{E} = 0, \ \mathbf{E} = \varepsilon \mathbf{E}, \ \mathbf{E} = -\nabla \phi \Rightarrow \nabla^2 \phi = 0 \)

BC: \( \hat{n} \cdot \mathbf{E} = 0 \Rightarrow \hat{n} \cdot \nabla \phi = 0 \)

Normal: \( \hat{n} = \hat{r} \times \hat{z} \)

Tangential: \( t = \hat{r} \cos(\theta) \hat{r} - \hat{z} \sin(\theta) \hat{z} \)

\( \nabla \cdot \hat{n} = 0 \Rightarrow \alpha = -\rho \hat{r} \hat{r} \frac{dz}{d\theta} \)

\( I = \int_{0}^{2\pi} \int_{0}^{\lambda} \rho \sin \theta \ d\rho \ d\theta \)

\( V = \int_{0}^{\lambda} d\theta \int_{0}^{\rho} \frac{1}{2} \rho R^2 \sin^2 \theta \ d\rho \ d\theta \)

1b) \( \lambda >> R \)

\( \int_{0}^{2\pi} \int_{0}^{\rho} \rho R^2 \sin^2 \theta \ d\rho \ d\theta = \int_{0}^{2\pi} \int_{0}^{\rho} \frac{1}{2} \rho R^2 \sin^2 \theta \ d\rho \ d\theta \)

\( V = \frac{\pi \lambda}{4 R^2} \left( 1 - a^2 \right)^{\frac{3}{2}} \)

\( R = \frac{\lambda}{\pi a^2 \left( 1 - a^2 \right)^{\frac{3}{2}}} \)

\( \lambda >> R \) implies \( \frac{\partial \theta}{\partial \phi} \gg \frac{\partial \theta}{\partial \phi} \)

\( \frac{1}{r} \frac{\partial}{\partial r} \left( r \hat{r} \frac{\partial \phi}{\partial r} \right) \approx 0 \quad \phi = \phi(\theta) \)

\( \frac{1}{r} \frac{\partial}{\partial r} \left( r \hat{r} \frac{\partial \phi}{\partial \phi} \right) \approx \phi = \phi(\theta) \)

\( \nabla \cdot \nabla \phi = -\frac{1}{R} \frac{d}{dz} \left( R^2 \frac{\partial \phi}{\partial z} \right) = 0 \)

\( -\frac{d}{dz} \frac{\partial}{\partial z} \left( R^2 \frac{\partial \phi}{\partial z} \right) = 0 \Rightarrow R^2 \frac{\partial \phi}{\partial z} = \text{const} \)

\( \text{c.e.} \quad I_z = \frac{\pi}{4 R^2} \)

(1)
\[ (c) \quad \Delta \ll \lambda, R \]

Lowest order will be $\Delta = 0$ smooth

Cylindrical, $E_z = \text{const}$

$\phi_0 = -2 E_0$

First order

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{\partial \phi}{\partial z^2} = 0 \]

BC

$\hat{n} \cdot \nabla (\phi_0 \phi) \quad \hat{n} = \hat{r} - \frac{\partial \hat{r}}{\partial z}$

\[ - \frac{d}{dz} \hat{r} \cdot \nabla (-2 E_0) + \hat{r} \cdot \nabla \phi = 0 \]

$\phi_0$

\[ E_0 \frac{d^2 r}{d z^2} + \frac{\partial \phi}{\partial r} = 0 \]

\[ \frac{d}{dz} \frac{d^2 r}{d z^2} = 2 \frac{d r}{d z} = -k^2 \phi_0 \sin k z \]

\[ \frac{d \phi}{d r} = E_0 \frac{k \Delta ^2}{2 \Delta} \sin k z \]

\[ \phi = \phi_0 \sin k z \quad \nabla^2 \phi_0 + \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \phi}{\partial r} + k^2 \phi = 0 \]

\[ \hat{r} = A I_0 (kr) \quad \frac{d \hat{r}}{d z} = A k I_0'(kr) \]

\[ A k I_0'(kr) = E_0 \frac{k \Delta ^2}{2 \Delta} \]

\[ \phi_0(r, z) = E_0 \sin k z \frac{\Delta ^2}{2 \Delta} \frac{I_{0}(kr)}{I_{0}(2k r)} \]

Modified Bessel $I_0$
\[ I = \int_{0}^{R(\xi)} \pi r^2 dr \times \xi \quad \quad E_2 = E_{20} \frac{k \cos \theta}{2 \pi} \frac{I_{0}^{(1)}(\theta)}{I_{0}^{(1/2)}(\theta)} \]

\[ I = \int_{0}^{2 \pi} \frac{R^2(\xi)}{2} \cdot \frac{I_{0}^{(1)}(\theta)}{I_{0}^{(1/2)}(\theta)} \] 

Note from Abramowitz & Stegun p. 376

\[ q_{1,6,26} \]

\[ Z \cdot I_{0}(\theta) = \frac{d}{dt} Z \cdot I_{1}(\theta) \]

\[ I_{1}(\theta) = Z \cdot I_{0}(\theta) \]

\[ I = 2\pi \sigma E_{20} \left\{ \frac{R^2(\xi)}{2} - \frac{\Delta^2}{2\pi k} \cos \theta \right\} \int_{0}^{\frac{\pi}{2}} \left\{ \frac{1}{2} \cdot I_{0}(\theta) \left( \frac{d}{d\theta} I_{0}(\theta) \right) \right\} \]

\[ e_{\text{sub}} = kR \]

\[ Z \cdot I_{0}(\xi) = \frac{1}{Z} \int_{0}^{\frac{\pi}{2}} \frac{d}{d\theta} I_{0}(\theta) \]

\[ Z = 2\pi \sigma E_{20} \left\{ \frac{R^2(\xi)}{2} - \frac{\Delta^2}{2\pi k} \cos \theta \right\} = 2\pi \sigma E_{20} \frac{R^2}{2} \]

\[ I = \text{constant as it should be} \]

Next order will change \( R \),

Resistance
2. Use scalar magnetic potential $\psi(x,y,z)$, because $\vec{J}_{\text{free}} = 0$

$$\vec{H} = -\nabla \psi, \quad \vec{B} = \mu_0 (\vec{H} \times \hat{z}), \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \nabla^2 \psi = \nabla \cdot \vec{M} \quad \nabla \cdot \vec{M} = (\delta(x-hz) - \delta(x+hz)) \begin{cases} -\mu_0, & 0 < h < L_2 - W \\ \mu_0, & L_2 < h < 2L_2 \\ 0, & \text{otherwise} \end{cases}$$

$x(x,t)$ is periodic in $z$ with period $P$

$$x(x,t) = \sum_{n=-\infty}^{\infty} X_n(x) e^{ik_nz} \quad k_n = \frac{2\pi}{P}$$

$$\nabla \cdot \vec{M} = \sum_{n=-\infty}^{\infty} D_n \left( \delta(x-hz) - \delta(x+hz) \right) e^{ik_nz}$$

$$D_n = \left\{ \begin{array}{c} \frac{p}{P} e^{ik_nz} \quad \text{if } 0 < h < L_2 - W \\ \frac{p}{P} e^{-ik_nz} \quad \text{if } L_2 < h < 2L_2 \end{array} \right.$$
\[ \left( \frac{d^2}{dx^2} - k_n^2 \right) X_n(x) = \delta(x - \frac{h}{2}) \cdot \delta(x + \frac{h}{2}) \]

\[ X_n(x) = A \text{ sinh k}_n x \quad 0 \leq x < \frac{h}{2} \]

\[ X_n(x) = B \exp(-k_n x) \quad \frac{h}{2} < x \]

Continuity of \( X_n \) \quad \( A \text{ sinh k}_n h/2 = B \exp(-k_n h/2) \) \quad (k_n < 0)

Jump in \( X_n \) \quad \left. \frac{dX_n}{dx} \right|_{x = h/2^+} = D_n = -k_n B \exp(-k_n h/2) - k_n A \text{ cosh k}_n h/2

\[ D_n = -k_n (A \text{ sinh k}_n h/2 + A \text{ cosh k}_n h/2) = -k_n A e^{-k_n h/2} \]

\[ A = -\frac{D_n}{k_n} e^{-k_n h/2} \]

\[ X_n(x) = -\frac{D_n}{k_n} \exp(-k_n x/2) \text{ sinh k}_n (x + h/2) \quad 0 < x < h/2 \]

\[ X_n(x) = -\frac{D_n}{k_n} \text{ sinh k}_n (x - h/2) \exp(-k_n x) \quad \frac{h}{2} < x \]