Green's function in spherical coordinate

\[ \nabla^2 G(r, r') = -4\pi \delta(r - r') \]

\[ G = \frac{1}{|r - r'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}} \]

\[ G = \sum \left[ A_l r^l + B_l r^{-l-1} \right] P_l(\cos \theta), \quad |r| > |r'| \]

\[ \Rightarrow G = \sum_l P_l(\cos \theta) A_l r^l, \quad r < r' \]

\[ G = \sum_l P_l(\cos \theta) B_l r^{-l-1}, \quad r > r' \]

\[ G(\theta = 0) = \sum_l P_l(\theta) A_l r^l \]

\[ P_l(\theta) = 1 \quad \text{as defined.} \]

when \( \theta = 0 \), \( \vec{r} = r \hat{\theta} \)

\[ G_{\theta = 0} = \frac{1}{|r - r'|} = \frac{1}{|r - r|} = \frac{1}{r(r - r')} \] \( (r < r') \)

Recall \( \sum_{l=0}^{\infty} \lambda^l = \frac{1}{1 - \lambda} \)

\[ \Rightarrow G_{\theta = 0} = \frac{1}{r} \sum_{l=0}^{\infty} \left( \frac{r}{r'} \right)^l \]
\[ \frac{1}{r} \sum_{l=0}^{\infty} \left( \frac{r}{r'} \right)^l = \sum_{l} A_l r^l \]

\[ \Rightarrow A_l = \frac{1}{r^{l+1}} \Rightarrow G = \sum_{l} P_l(\cos \theta) \frac{r^l}{r'^{l+1}}, \quad (r < r') \]

Look at this:

Ex. charged ring:

\[ \phi(z) = \frac{q}{4\pi \varepsilon_0 \sqrt{z^2 + a^2}} \]

1. Using Taylor expansion, \( z < a \):

   \( l \) is even:

   \[ \frac{1}{\sqrt{z^2 + a^2}} = \sum_{l} P_l(0) \frac{z^l}{a^{l+1}}, \quad z < a \]

2. \( l \) is odd:

   \[ \frac{1}{\sqrt{z^2 + a^2}} = \sum_{l} P_l(0) \frac{a^l}{z^{l+1}}, \quad z > a \]

So:

\[ G = \frac{1}{|r - r'|} = \sum_{l} P_l(\cos \theta) \frac{r^l}{r'^{l+1}} \]
Laplace's Equation in a conductor with finite conductivity

Look at the egg plant below:

\[ \nabla \cdot \mathbf{E} = \frac{\partial \Phi}{\partial t}, \quad \Phi = \text{potential} \]

conditions: \( \frac{\partial \Phi}{\partial x_n} = 0 \) on the boundary

How does the current distribute itself?

What is the resistance of the egg plant?

\( \mathbf{J}(\mathbf{r}) \) is the current density, unit \( \frac{\text{A}}{\text{m}^2} \).

\[ \mathbf{J} = \int_S \mathbf{d}a \cdot \mathbf{J} \]

Point form of Ohm's Law:

\[ \mathbf{J} = \sigma \mathbf{E} \]

\( \sigma \) = electrical conductivity.
From KCL: \[ \oint \text{da} \ n \cdot \overline{J} = 0 \]

Use divergence theory: \[ \nabla \cdot \overline{J} = 0 \]

\[ \frac{dQ}{dt} + I = 0. \]

From KVL: \[ \oint \text{dl} \cdot \overline{E} = 0 \Rightarrow \overline{E} = -\nabla \phi \]

We are allowed to forget these formulas, but we should know how to get them from KCL and KVL.

• Some derivation:

\[ \overline{J} = -\sigma \nabla \phi \]

\[ \Rightarrow \nabla \cdot \overline{J} = -\nabla \cdot (\sigma \nabla \phi) = 0 \]

\[ \Rightarrow - (\nabla \sigma \cdot \nabla \phi + \sigma \nabla^2 \phi) = 0. \]

If \( \sigma \) is a function of \( \mathbf{r} \)

\[ \nabla^2 \phi = -\frac{\nabla \sigma \cdot \nabla \phi}{\sigma} \]

If \( \sigma \) is a constant:

\[ \sigma \nabla^2 \phi = 0 \Rightarrow \nabla^4 \phi = 0 \]
Easy example:

We stick to food for examples. Let's look at a cube of solid, uniform cheese:

\[ \nabla^2 \phi = 0 \]

Boundary:
\[
\begin{align*}
  z = 0, & \quad \phi = V \\
  z = L, & \quad \phi = 0.
\end{align*}
\]

\[
\frac{\partial^2 \phi}{\partial z^2} = 0 \quad \Rightarrow \quad \phi = A + Bz = \frac{L - z}{L} V
\]

\[
E_z = -\frac{\partial \phi}{\partial z} = \frac{V}{L}
\]

\( \Rightarrow \) too easy.
Laplace's equation example.

Let's look at the same problem, but this time the cheese has a spherical void inside.

The currents will try to avoid the void.

As \( |r| \to \infty \), the electric field will be constant.

\[ E \to E_0 \frac{1}{r} \]

\[ \nabla^2 \phi = 0 , \ |r| > a , \ \text{because we assume constant } \sigma \]
Two boundaries:

\[ |r| \to \infty \quad \Rightarrow \quad \phi \to -E_0 z \quad (-\nabla \phi \to \nabla E_0) \]

\[ |r| = a \quad \Rightarrow \quad \frac{\partial \phi}{\partial r} \bigg|_{r=a} = 0 \quad \text{(no normal E component)} \]

and \( z E_0 = r \cos \theta E_0 \)

We have:

\[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \phi + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \phi = 0 \]

Solve this:

\[ \phi = \sum_{l=0}^{\infty} \left[ A_l \, r^l + B_l \, r^{-l-2} \right] P_l (\cos \theta) \]

as \( r \to \infty \)

\[ \phi = \sum_{l} A_l \, r^l \, P_l (\cos \theta) = -r \cos \theta \, E_0 \]

When \( l = 1 \), \( P_1 (\cos \theta) = \cos \theta \) by definition of Legendre polynomial.

\( \Rightarrow \) Must have \( \int \) \( A_l = 0 \), \( l \neq 1 \)

\[ A_1 = -E_0, \quad l = 1 \]

\[ \Rightarrow \quad \phi (r) = A_1 r \cos \theta + \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l (\cos \theta) \]
At \( r=a \):

\[
\frac{\partial \phi}{\partial r} \bigg|_{r=a} = A_1 \cos \theta - \sum \frac{B_l (l+1)}{a} \frac{P_l (\cos \theta)}{a^{l+2}} = 0
\]

- \( l=1 \Rightarrow A_1 - \frac{B_1}{a^3} = 0 \) and \( A_1 = -E_0 \)
  \Rightarrow B_1 = -E_0 \frac{a^3}{2}

- \( l \neq 1 \) : \[
\frac{B_l (l+1)}{a^{l+2}} = 0 \Rightarrow B_l = 0
\]

\[
\phi (r) = -E_0 \cos \theta \left( r + \frac{a^3}{2r^2} \right) \quad (r > a)
\]

- What is the potential inside the void? \((r < a)\)

Conditions:

\[\nabla^2 \phi = 0\]

and \( \phi \) is continuous at the boundary.

As \( r \to a \) inside:

\[
\phi \to -E_0 \cos \theta \left( \frac{3}{2} \right) a
\]

Let \( \phi = \sum \left( C_l r^l + D_l r^{-(l+1)} \right) P_l (\cos \theta) \)
As \( r \to 0 \), \( r^{(l+1)} \) blows up

\[ \Rightarrow D_0 \text{ has to be 0}. \]

As \( r \to a \):

\[ \phi \to \sum_{l} C_{l} a^l \phi_e \cos(\theta) \]

\[ \begin{cases} 
  C_{l} = 0, & l \neq 1. \\
  C_{1} a = -\frac{3}{2} \frac{F_{0} a}{l} \Rightarrow C_{1} = -\frac{3}{2} \frac{F_{0}}{l} 
\end{cases} \]

\[ \Rightarrow \boxed{\phi(r) = -\frac{3}{2} \frac{r \cos \theta F_{0}}{l}} \quad \text{inside.} \quad (r < a) \]

Electric field is constant inside.