7. Poisson and Laplace Equation.

\[ \nabla \cdot \overline{E} = \frac{f}{\varepsilon_0} \]

\[ \overline{E} = -\nabla \phi \]

\[ \Rightarrow \nabla^2 \phi = -\frac{f}{\varepsilon_0} \rightarrow \text{Poisson equation.} \]

In a space that lacks a charge density, the scalar potential satisfies Laplace equation:

\[ \nabla^2 \phi = 0 \]

Two useful Relations for Dirac-Delta function:

\[ \nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) = -\frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \]

\[ \nabla \cdot \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) = \nabla \cdot \left[ -\frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \right] = -4\pi \delta (\vec{x} - \vec{x}') \]
8. Boundary Conditions

From book: If a surface $S$ has a surface charge density $\sigma(x)$ and electric fields $\vec{E}_1$ and $\vec{E}_2$ on either side, then

$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}$$

Boundary Conditions on Surfaces:

$$\phi(\vec{x}) = \int d^3x' \frac{\sigma(x')}{|\vec{x} - \vec{x}'|}$$

$$E = -\nabla \phi(\vec{x}) \Rightarrow \oint E \cdot dl = 0$$

represents a complete solution for the electric field given the charge density. Problem is we do not always know the charge density before hand, but we must sometimes determine it as part of the solution of the problem.

For example, suppose we introduce a perfect electrical conductor into our problem. Charge on such a conductor is free to move and arrange itself
Such that $\overline{E} = \overline{0}$ within the conductor.

Once $\overline{E} = \overline{0}$, there is no longer any electrical force on a free charge within the conductor, inducing the charge to move.

(This does not imply that there is no force on the conductor as we shall see.)

\[ \nabla \cdot \overline{E} = 0 \]

In conductor, $\nabla \cdot \overline{E} = 0$

$\Rightarrow$ no net charge density inside conductor.

- At boundary:

\[ \oint \overline{E} \cdot d\overline{l} = 0 \]

\[ \text{tangential component: } \overline{E}_t - \overline{E}_{to} = \overline{0} \]

but $\overline{E}_{to} = 0$ (inside conductor) $\Rightarrow$ $\overline{E}_t = 0$

$\phi_1 - \phi_2 = \int_1^2 \overline{E} \cdot d\overline{l} = 0$

$\Rightarrow \phi = \text{const}$
Normal component:

\[ \varepsilon_0 \int da \, \vec{m} \cdot \vec{E} = q_{\text{enclosed}} = \int \sigma \, da \]

\[ \vec{E} = \vec{0} \text{ inside} \]

\[ \Rightarrow \vec{m} \cdot \vec{E} = \frac{1}{\varepsilon_0} q_{\text{enclosed}} = \frac{\sigma}{\varepsilon_0} \]

An Example

Suppose we move a perfect conductor into the picture.

What we want to solve is:

\[ \varepsilon_0 \nabla^2 \phi = -\rho(x) \quad \text{in vacuum} \]

\[ \phi \text{ is specified on surface: "Dirichlet" condition} \]

\[ \vec{n} \cdot \nabla \phi = \frac{\partial \phi}{\partial n} \text{ specified: "Neumann" condition} \]
Some examples:

Vacuum Diode:

One dimension problem

\[
\frac{d^2 \phi}{dx^2} = -\frac{P(x)}{\varepsilon_0}
\]

Find \( \phi \)

We solve this using integration:

\[
\frac{d\phi}{dx} = \frac{d\phi}{dx} \bigg|_{x=0} - \int_0^x \frac{dx'}{\varepsilon_0} P(x')
\]

\( \phi(x) = \phi \bigg|_{x=0} + \int_0^x \frac{dx'}{\varepsilon_0} \int_0^x \frac{dx''}{\varepsilon_0} P(x'') \)

Changing variables:

\[
\int_0^x dx' \int_0^{x'} \frac{dx''}{\varepsilon_0} P(x'') = \int_0^x dx' \int_0^{x} \frac{dx''}{\varepsilon_0} P(x'')
\]

\[
= \int_0^x dx' \frac{P(x')}{\varepsilon_0} (x-x')
\]
Boundary condition: \( \phi(d) = V_0 = d \left. \frac{d\phi}{dx} \right|_{x=x_0} - \int_0^d dx'' \frac{\rho(x'')}{\varepsilon_0} (d-x'') \)

\[
\left. \frac{d\phi}{dx} \right|_{x=0} = \frac{V_0}{d} + \int_0^d dx'' \frac{\rho(x'')}{\varepsilon_0} (d-x'')
\]

Combine all

\[
\Rightarrow \phi(x) = \frac{x}{d} V_0 + \int_0^d dx'' \frac{\rho(x'')}{\varepsilon_0} (d-x'') x
\]

Consider the last term:

\[
\Rightarrow \int_0^d dx'' \frac{\rho(x'')}{\varepsilon_0} \times \begin{cases} \ 0 & \text{if } x'' > x \\ (x-x'') & \text{if } x'' < x \end{cases}
\]

\[
\Rightarrow \phi(x) = \frac{x}{d} V_0 + \int_0^d dx'' \frac{\rho(x'')}{\varepsilon_0} G(x, x'')
\]

Green's function:

\[
G = \begin{cases} \ x (1-\frac{x''}{d}) & \text{if } x'' > x \\ x'' (1-\frac{x}{d}) & \text{if } x'' < x \end{cases}
\]

x: point where you are seeing

x'': point of charge
Motion example:

A vacuum diode is constructed as shown.

We have:

\[ J = \text{current density} = \frac{I}{S} = \text{const} \]

\[ J = n(x) \cdot q \cdot v_x \]

**Constant of motion:**

\[ \frac{1}{2} m v_x^2(x) - e \phi(x) = \text{const.} \]

At the cathod: \( \text{const} = 0 \), \( (v_x(0) = 0, \phi(0) = 0) \)

\[ \Rightarrow \frac{1}{2} m v_x^2(x) = e \phi(x) \]

\[ v_x = \sqrt{\frac{2e\phi(x)}{m}} \]

With current density \( J_x = \text{const} \)

\[ f(x) = -\frac{J_x}{v_x} \]

\[ \Rightarrow \frac{d^2\phi}{dx^2} = \frac{J_x}{\varepsilon_0 \sqrt{\frac{2e}{m}}} \]
Current density depends on emission rate and electric field at cathode

A) Thermal emission limited: \( J \) depends on \( T \)

B) Space charge limited emission:

\( J_x \) adjusts itself to make \( E_x(0) = -\frac{d\phi}{dx}\bigg|_0 = 0 \)

Solution with:

\[
\begin{cases}
\phi(0) = 0 \\
\frac{d\phi(x)}{dx} = 0
\end{cases}
\]

\[
\phi = \left[ \frac{3}{4} \left( \frac{4I}{\varepsilon_0} \right)^{1/2} \left( \frac{m}{2e} \right)^{1/4} \right]^{4/3} \times \frac{4}{3}
\]

Applying \( \phi(d) = V_0 \):

\[
J = \frac{4}{9} \varepsilon_0 \left( \frac{2e}{m} \right)^{1/2} \frac{V_0^{3/2}}{d^{1/2}}
\]

\[
I = MV_0^{3/2}, \quad M = \frac{4A\varepsilon_0}{9 \frac{d^3}{d^2}} \left( \frac{2e}{m} \right)^{1/2}
\]

\( M \) is called perveance.
9. Green's Theorem

and

Green's Function

Recall Green's function:

\[ G(x, x') = \frac{1}{|x - x'|} \quad \text{in 3D} \]

Solution to Poisson's Equation is

\[ \phi(x) = \frac{1}{4\pi \varepsilon_0} \int d^3\alpha \frac{P(\alpha)}{|x - x'|} \]

\[ \Leftrightarrow \nabla^2 \phi = \frac{1}{4\pi \varepsilon_0} \int d^3\alpha \frac{P(\alpha) \nabla^2 1}{|x - x'|} \]

\[ \Leftrightarrow -\frac{P(x)}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int d^3\alpha P(\alpha') \left[ \nabla^2 \frac{1}{4\pi |x - x'|} \right] \]

\[ \Leftrightarrow \nabla^2 \left( \frac{1}{|x - x'|} \right) = -4\pi \delta(x - x') \]

\[ \Leftrightarrow \nabla^2 G(x, x') = -4\pi \delta(x - x') \]
To derive Green's identities, we start with the divergence theorem:

\[ \int_V d^3x \nabla \cdot \mathbf{A} = \oint_S \mathbf{A} \cdot \mathbf{n} \, ds \]

Special case: \( \mathbf{A} = \phi \nabla \psi \)

\[ \nabla \cdot \mathbf{A} = \nabla \cdot (\phi \nabla \psi) = \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi \]

\[ \Rightarrow \int_V d^3x (\nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi) = \oint_S \phi \nabla \psi \cdot \mathbf{n} \, ds \]

**Green's first identity**

Vice versa: \( \int_V d^3x (\nabla \psi \cdot \nabla \phi + \psi \nabla^2 \phi) = \oint_S \psi \nabla \phi \cdot \mathbf{n} \, ds \)

Subtract them: This is Green's second identity.

\[ \int_V d^3x (\phi \nabla^2 \psi - \psi \nabla^2 \phi) = \oint_S ds \mathbf{n} \cdot (\phi \nabla \psi - \psi \nabla \phi) \]

This is Green's first identity: derived above.

\[ \int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d^3x = \oint_S \phi \nabla \psi \cdot \mathbf{n} \, da \]
Example: How to use Green's Theorem?

\[ \nabla^2 \phi = -\frac{\rho(\vec{x})}{\varepsilon_0}, \]

\[ \nabla^2 \psi = -4\pi \delta(\vec{x} - \vec{x}^*), \]

Using Green's second identity:

\[ \int_V d^3x \phi(\vec{x}) [-4\pi \delta(\vec{x} - \vec{x}^*)] + \int_V d^3x \psi \frac{\rho(\vec{x})}{\varepsilon_0} = \int_S d\vec{m} \cdot (\phi \nabla \psi - \psi \nabla \phi), \]

\[ \Leftrightarrow -4\pi \phi(\vec{x}^*) + \int_V d^3x \psi \frac{\rho(\vec{x})}{\varepsilon_0} = \int_S d\vec{m} \cdot (\phi \nabla \psi - \psi \nabla \phi). \]

**Case A**: Dirichlet condition, \( \psi = 0 \) on the boundary

\[ \Rightarrow \phi(\vec{x}^*) = \frac{1}{4\pi} \left[ \int_V d^3x \frac{\rho(\vec{x})}{\varepsilon_0} \psi(\vec{x}, \vec{x}^*) - \int_S d\vec{m} \cdot \nabla \psi \right]. \]

**Case B**: Neumann condition, let \( m \cdot \nabla \psi = K |_S \)

- First, do solution exist?
  - Not necessarily.

From Gauss's Law

\[ \varepsilon_0 \int_S d\vec{a} \cdot \vec{E} = -\varepsilon_0 \int_S d\vec{a} \cdot \vec{m} \cdot \nabla \phi = q, \]

with \( q = \int_V d^3x \rho(\vec{x}) \).
In order for a solution to exist, one must specify values of $\nabla \phi$ that are consistent with Gauss's Law.

- Second, if a solution exists, then is it unique?
  - No, it is not.

Suppose $\phi(x)$ satisfies equations and boundary cond.

Then $\phi(x) + \text{const}$ also satisfies those.

So, to make a solution unique, we must add an additional condition.

\[ \text{E.g.: } \int_S \nabla \phi = 0 \]

The average value of $\phi$ on the surface is zero.

... Back to the example before.

Gauss's Law

\[ \Rightarrow \int_S \nabla \phi \cdot dA = -4\pi \int_V \rho(x) dV \]

\[ \Rightarrow K = \frac{-4\pi}{A} \]
10. Uniqueness of Solution. 

Consider
\[ \nabla^2 \phi = -\frac{f}{\varepsilon^2} \]

Subject to boundary conditions on enclosing surface. Suppose we have found a solution satisfying the equation and all boundary conditions. Is that solution unique? Ans: Yes.

Proof: Use contradiction.

Suppose \( \phi_1 \) and \( \phi_2 \) are two different solutions then
\[ U = \phi_1 - \phi_2 \text{ satisfies } \nabla^2 U = 0 \]

with BC's
\[
\begin{cases} 
U = 0 & \text{if } \phi \text{ satisfies Dirichlet} \\
\partial U/\partial n = 0 & \text{if } \phi \text{ satisfies Neumann}
\end{cases}
\]

\[
\int d^3 x \left( \nabla^2 U \right) = \int d^3 x \left[ (\nabla \cdot (n \nabla U)) - (\nabla U)^2 \right] = 0
\]

Thus
\[
\int d^3 x |\nabla U|^2 = \int d a (n \cdot \nabla U) U
\]

If either \( U \) or \( n \cdot \nabla U = 0 \) on \( S \) then we must have
\[
\int d^3 x |\nabla U|^2 = 0 \Rightarrow \nabla U = 0
\]
\[ U = \text{const} \]

for Dirichlet: \[ U = 0 \]

for Neumann: constant is unimportant

From this, it follows that we cannot specify both \[ \phi \text{ and } m \nabla \phi \]

\[ \Rightarrow \text{ The solution is unique, if it exists.} \]
Example: Green's function in one dimension

\[
\frac{d^2 G_0(x, x^{'})}{dx^2} = -4\pi \delta(x - x^{'})
\]

\[
G_0(x, 0) = G_0(x, b) = 0
\]

for \(0 < x^{'}, x < b\):

\[
G_0 = k_1 x^{'}, \quad G_0 = k_2 (x - b)
\]

at \(x = x^{'},\) \(G_0(x, x^{'})\) is continuous.

\[
\Rightarrow k_1 x = k_2 (x - b)
\]

Also, \(\int_{x - \epsilon}^{x + \epsilon} dx^{' \epsilon} d^2 G_0 = \frac{d}{dx^{'}} G_0 \bigg|_{x - \epsilon}^{x + \epsilon} = -4\pi \)

\[
\Rightarrow \begin{cases} 
  k_2 - k_1 = -4\pi \\
  k_1 x = k_2 (x - b) 
\end{cases} \Rightarrow \begin{cases} 
  k_2 = -4\pi \frac{x}{b} \\
  k_1 = 4\pi \left(1 - \frac{x}{b}\right) 
\end{cases}
\]

\[
\Rightarrow G_0(x, x^{'}) = \begin{cases} 
  4\pi \left(1 - \frac{x}{b}\right) x^{'}, \quad 0 < x^{'}, x < b \\
  4\pi \frac{x^{'}}{b}, \quad x < x^{'}, b
\end{cases}
\]