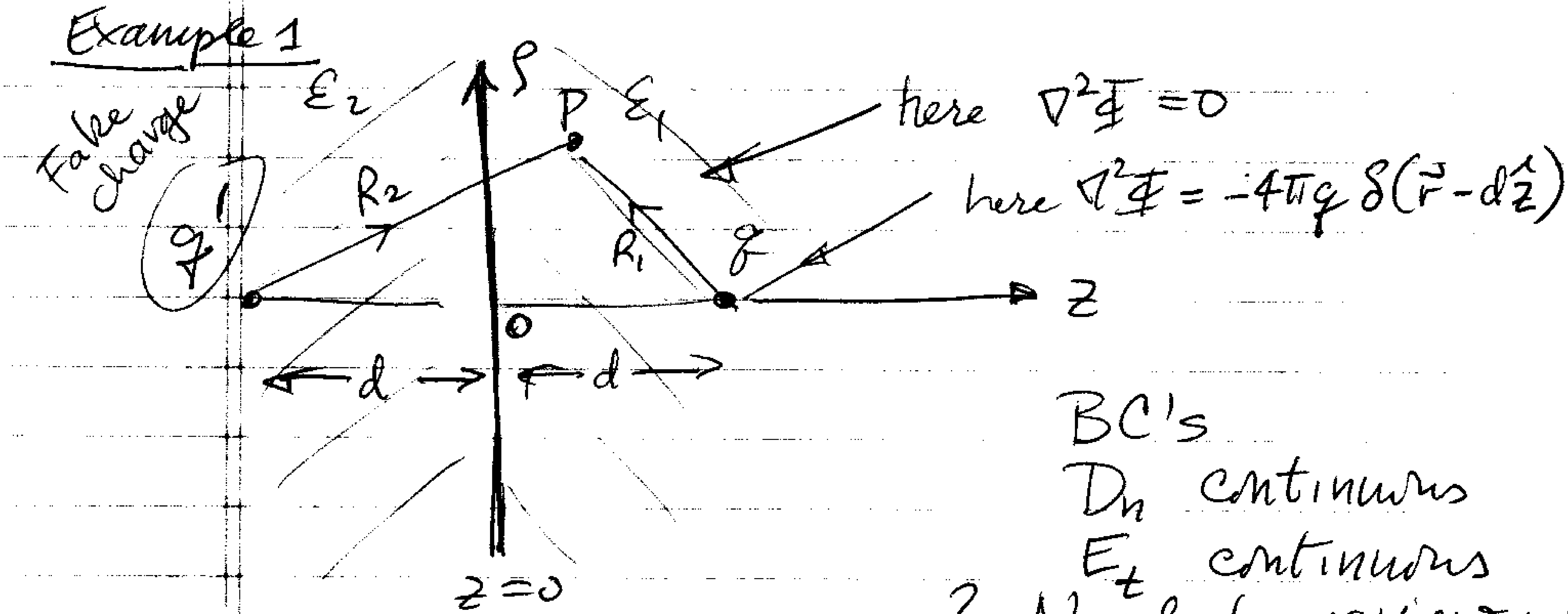
(3106)

Boundary Value Problems for Dielectrics (1)

Example 1



BC's
D_n continuous
E_t continuous
? Need to review why?

ASK:
(does class know method of images for conductors?)

Use "Method of Images" for Dielectrics

Put effective charge q' at $z=-d$, $\rho=0$

As in method of images for conductors, role of q' is to provide satisfaction of BC's at $z=0$

If constructed solution Φ using q' satisfies $\begin{cases}\nabla^2\Phi_1 = 0\\ \nabla^2\Phi_1 = -4\pi\delta(\vec r - d\hat z)\end{cases}$ and BC's, then Φ is unique.

$$\Phi_1(z>0) = \frac{q}{\epsilon_1 R_1} + \frac{q'}{\epsilon_1 R_2}$$

use ϵ_1 here, it will just affect value for q' we get later

$$R_1 = \sqrt{\rho^2 + (z-d)^2} \qquad R_2 = \sqrt{\rho^2 + (z+d)^2}$$

But unlike conductor case we now must also find $\Phi(z<0)$ $= \Phi_2$ — why?

For $z<0$, try the reasonable guess $\Phi_2 = \dfrac{q''}{\varepsilon_2 R_1}$

q'' to be determined

ε_2: in material 2

R_1: dist from q

Now, apply BC's :

$$\epsilon_2 \frac{\partial \Phi_2}{\partial z}\Big)_{z=0} = \epsilon_1 \frac{\partial \Phi_1}{\partial z}\Big)_{z=0}$$

Now $\frac{\partial}{\partial z} \frac{1}{R_1}\Big)_{z=0} = \frac{-1}{R_1^2} \frac{\partial R_1}{\partial z}\Big)_{z=0} = \frac{-1}{R_1^2} \frac{1}{2} \frac{1}{R_1} \; 2(z-d)\Big)_{z=0} = \frac{d}{R_1^3}\Big)_{z=0}$

$$= \frac{d}{(\rho^2+d^2)^{3/2}}$$

and $\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)_{z=0} = \frac{-d}{(\rho^2+d^2)^{3/2}}$

$$\Rightarrow \epsilon_2 \frac{q''}{\epsilon_2} \frac{d}{(\rho^2+d^2)^{3/2}} = \epsilon_1 \left(\frac{q}{\epsilon_1} \frac{d}{(\rho^2+d^2)^{3/2}} + \frac{q'}{\epsilon_1} \frac{(-d)}{(\rho^2+d^2)^{3/2}} \right)$$

or $q'' = q - q'$ (1)

<u>E_t continuous</u>

$$E_\rho(z=0^-) = E_\rho(z=0^+)$$

$$\frac{\partial \Phi_2}{\partial \rho}\Big)_{z=0} = \frac{\partial \Phi_1}{\partial \rho}\Big)_{z=0}$$

Now $\frac{\partial}{\partial \rho} \frac{1}{R_1}\Big)_{z=0} = -\frac{1}{R_1^2} \frac{\partial R_1}{\partial \rho}\Big)_{z=0} = -\frac{1}{R_1^2} \frac{1}{2} \frac{1}{R_1} 2\rho\Big)_{z=0}$

$$= \frac{-\rho}{(\rho^2+d^2)^{3/2}}$$

and also $\frac{\partial}{\partial \rho} \frac{1}{R_2}\Big)_{z=0} = \frac{-\rho}{(\rho^2+d^2)^{3/2}}$

$$\Rightarrow \frac{q''}{\epsilon_2} \frac{(-\rho)}{(\rho^2+d^2)^{3/2}} = \frac{q}{\epsilon_1} \frac{(-\rho)}{(\rho^2+d^2)^{3/2}} + \frac{q'}{\epsilon_1} \frac{(-\rho)}{(\rho^2+d^2)^{3/2}}$$

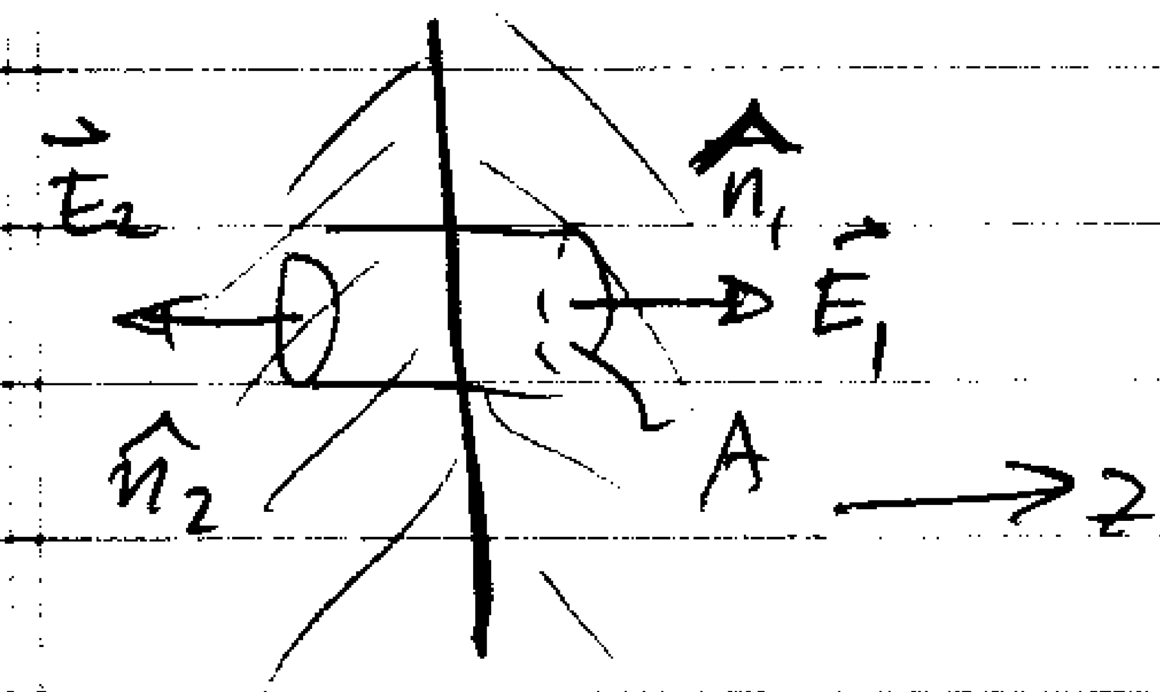
$$\Rightarrow q''/\epsilon_2 = \frac{1}{\epsilon_1}(q+q') \quad (2)$$

Combine ① and ② to give

$$q' = -\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\right) q$$

$$q'' = \left(\frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\right) q$$

What is the surface charge density at $z=0$?



$$4\pi\sigma A = \left(\vec{E}_1 \cdot \hat{n}_1 + \vec{E}_2 \cdot \hat{n}_2\right) A$$

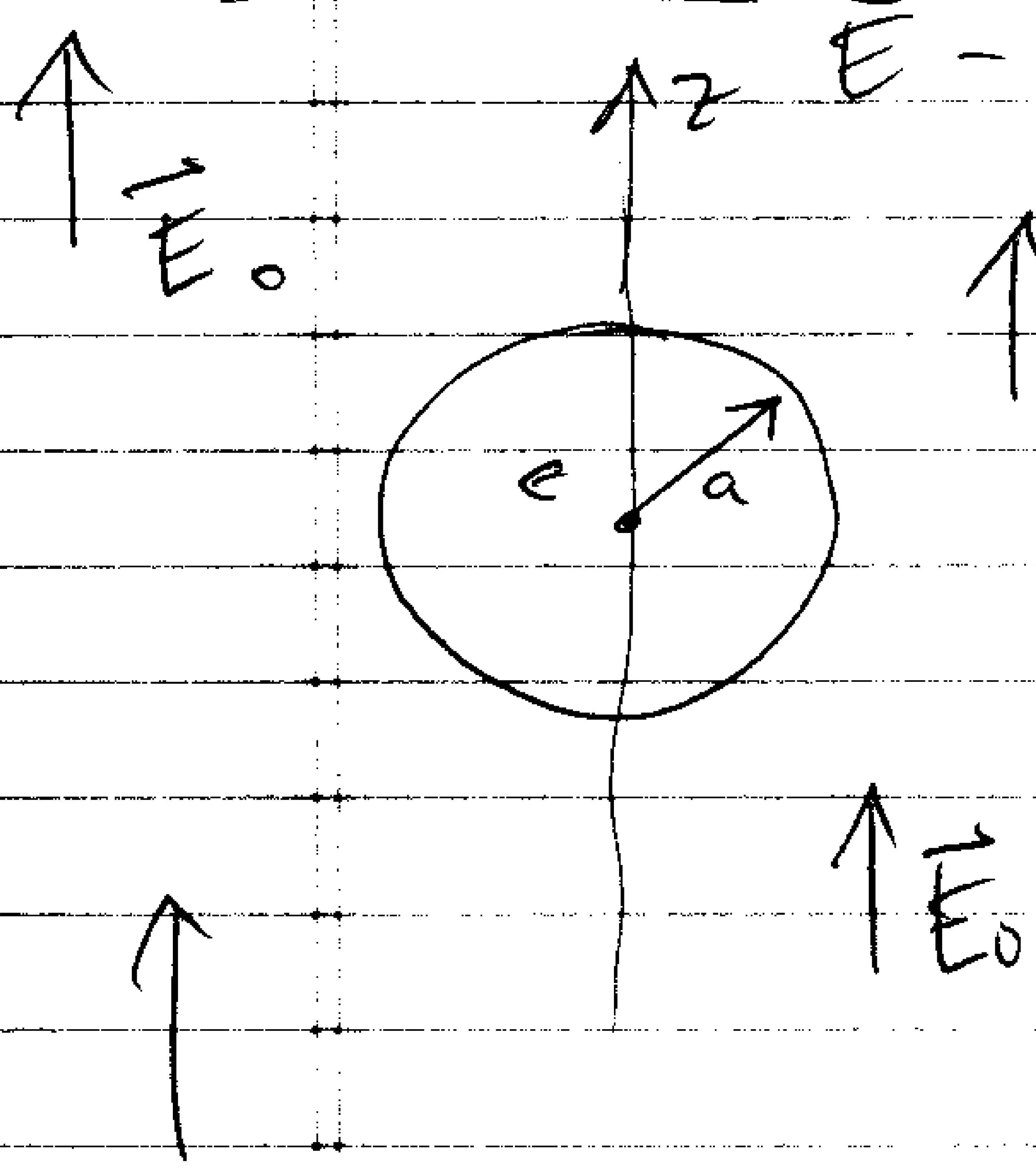
or $E_{1z} - E_{2z} = 4\pi\sigma$

$$\sigma = \frac{1}{4\pi}\left(\left(-\frac{\partial \Phi_1}{\partial z}\right)_{z=0} + \left(\frac{\partial \Phi_2}{\partial z}\right)_{z=0}\right)$$

or $$\sigma = \frac{1}{4\pi}\left(-\frac{q}{\epsilon_1}\frac{d}{(\rho^2+d^2)^{3/2}} + \frac{q'}{\epsilon_1}\frac{d}{(\rho^2+d^2)^{3/2}} + \frac{q''}{\epsilon_2}\frac{d}{(\rho^2+d^2)^{3/2}}\right)$$

$$= \frac{q}{2\pi}\,\frac{(\epsilon_2-\epsilon_1)}{\epsilon_1(\epsilon_2+\epsilon_1)}\,\frac{d}{(\rho^2+d^2)^{3/2}}$$

<u>Example 2</u> Dielectric sphere embedded in uniform $\vec{E}$ - field



<u>ϕ - symmetry</u>

$$r<a \quad \Phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$r>a \quad \Phi_{out} = \sum_{l=0}^{\infty}\left(B_l r^l + C_l r^{-(l+1)}\right) \cdot P_l(\cos\theta)$$

At $\vec{r} \to \infty$ $\quad -\nabla \Phi = E_0 \hat{z}$

or $\dfrac{\partial \Phi}{\partial z} = -E_0$

$\Rightarrow \quad \Phi(\vec{r}) = -E_0 z + \text{const} \quad (\text{const} \to 0)$

or $\Phi(\vec{r}) = -E_0 r\cos\theta \Rightarrow B_1 = -E_0$

<u>BC's</u> <u>D_n cont.</u> $\quad \epsilon \dfrac{\partial \Phi_{in}}{\partial r}\Big)_{r=a} = \dfrac{\partial \Phi_{out}}{\partial r}\Big)_{r=a}$

<u>E_t cont</u> $\quad \dfrac{1}{a}\dfrac{\partial \Phi_{in}}{\partial r}\Big)_{r=a} = \dfrac{1}{a}\dfrac{\partial \Phi_{out}}{\partial r}\Big)_{r=a}$

<u>D_n cont</u> $\quad \displaystyle\sum_{l=0}^{\infty}\left(l A_l a^{l-1} - l B_l a^{l-1} + (l+1) C_l a^{-(l+2)}\right) P_l = 0$

for all l

$\Rightarrow \quad A_1 - B_1 + C_1/a^3 = 0$

$\Rightarrow \quad A_1 = -E_0 + C_1/a^3 \to ①$

and $\quad A_l = \dfrac{C_l}{a^{2l+1}} \quad$ for $l \neq 1$

<u>E_t cont</u>
(you do)

$\epsilon A_1 = -E_0 - 2\dfrac{C_1}{a^3} \to ②$

$\epsilon l A_l = -(l+1)\dfrac{C_l}{a^{2l+1}} \quad$ for $l \neq 1$

compare

$\to$ only satisfied for $A_l = C_l = 0$ for $l \neq 1$

Solve ①, ② to give $A_1 = -\left(\dfrac{3}{2+\epsilon}\right) E_0$

$C_1 = \left(\dfrac{\epsilon-1}{\epsilon+2}\right) a^3 E_0$

$\Rightarrow \Phi_{in} = -\left(\dfrac{3}{\epsilon+2}\right) E_0 r\cos\theta = -\left(\dfrac{3}{\epsilon+2}\right) E_0 z$

$\Phi_{out} = -E_0 r\cos\theta + \left(\dfrac{\epsilon-1}{\epsilon+2}\right) E_0 \dfrac{a^3}{r^2}\cos\theta$

dipole potential why?

Inside the sphere

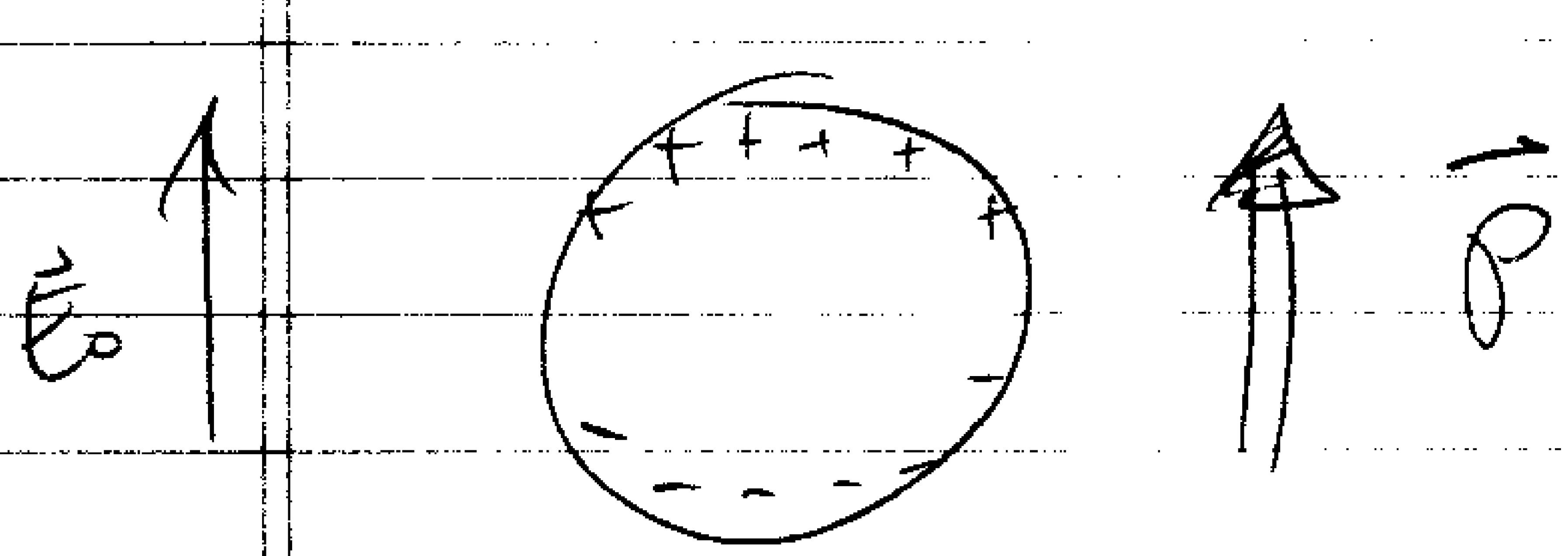
$$\vec{E}_{in} = -\nabla\Phi = -\hat{z}\,\frac{\partial\Phi}{\partial z} = \left(\frac{3}{\epsilon+2}\right) E_0 \hat{z}$$

$< E_0$ if $\epsilon > 1$
(why?)

can write

$$\Phi_{out} = -E_0 z + \frac{\vec{p}\cdot\hat{r}}{r^2}$$

where $\vec{p} = \left(\frac{\epsilon-1}{\epsilon+2}\right) a^3 E_0 \hat{z}$ induced dipole moment of dielectric sphere.



we will use this result in 681 when we do EM wave scattering from dielectric particles