Chapter 24

Maxwell's Displacement Current

$$\nabla \times \underset{\sim}{E} = - \frac{\partial \underset{\sim}{B}}{\partial t} \qquad \text{Faraday}$$

$$\nabla \cdot \underset{\sim}{E} = \cancel{4\pi}\rho / \epsilon_0 \qquad \text{Poisson}$$

$$\nabla \cdot \underset{\sim}{B} = 0 \qquad \text{Gauss}$$

$$\nabla \times \underset{\sim}{B} = \overset{\mu_0}{\cancel{\frac{4\pi}{c}}} \underset{\sim}{J} + ? \qquad \text{Ampere}$$

Take divergence of Ampere

$$\nabla \cdot \nabla \times \underset{\sim}{B} = 0 = \mu_0 \nabla \cdot \underset{\sim}{J}$$

for any $\underset{\sim}{B}$

$\nabla \cdot \underset{\sim}{J} = 0$ only for static currents

for time varying currents

$$\nabla \cdot \underset{\sim}{J} = - \frac{\partial \rho}{\partial t} \qquad \text{continuity equatn}$$

$$= - \frac{\partial}{\partial t} \epsilon_0 \nabla \cdot \underset{\sim}{E}$$

thus,

$$\nabla\cdot\left(\mu_0\left(\underset{\sim}{J}+\frac{\partial}{\partial t}\epsilon_0 E\right)\right)$$

$$\nabla\cdot\left(\nabla\times\underset{\sim}{B}\right) \overset{=0=}{=} \nabla\cdot\left[\cancel{\frac{4\pi}{c}}\,\mu_0\,\underset{\sim}{J}+\frac{1}{c}\frac{\partial E}{\partial t}\right]$$

THERE FORE IT IS REASonable to assume

$$\nabla\times\underset{\sim}{B} = \mu_0\,\cancel{\frac{4\pi}{c}}\left(\underset{\sim}{J}+\underbrace{\epsilon_0\frac{\partial E}{\partial t}}\right)$$

Maxwell's displacement current.

not a current

discuss integral version

Energy Conservation.

Suppose a configuration of charges and currents are set up in a system

$\rho(t,\underset{\sim}{x})$ $\quad\quad$ $\underset{\sim}{J}(\underset{\sim}{x},t)$

Macroscopic Media

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0$$

$$\nabla \cdot E = (\rho_f + \rho_i)/\epsilon_0$$

$$\nabla \times B = \mu_0 (J_f + J_m + J_p) + \epsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

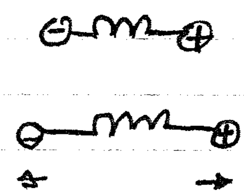
$$\nabla \times \left(\frac{B}{\mu_0} - M\right) = J_f$$

$$\rho_i = -\nabla \cdot P \qquad \frac{\partial \rho_i}{\partial t} + \nabla \cdot \frac{\partial P}{\partial t} = 0$$

continuity of induced charge

$$\frac{\partial P}{\partial t} = J_p =$$ current associated with changing of electric dipole strength

P



$P \to P' \quad \left(\frac{\partial P}{\partial t} t\right)$

small current $\propto \frac{\partial P}{\partial t}$

$$\cancel{\int d^3k \, \frac{e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x} - kc(t-t')}}{|\mathbf{x}-\mathbf{x}'|}}$$

~~steady state GF~~

$$\cancel{-k^2 G = -4\pi}$$

$$\cancel{G(k) = \frac{4\pi}{k^2}}$$

$$\frac{d}{dt}\left(\mathbf{P}_{mech} + \mathbf{P}_{fields}\right) = \int dA \, \mathbf{n}\cdot\underline{\underline{T}}$$

$$\underline{\underline{T}} = \left[\epsilon_0 \mathbf{E}\mathbf{E} + \mu_0 \mathbf{H}\mathbf{H} - \underline{\underline{1}}\,\frac{1}{2}\left(\epsilon_0|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right)\right]$$

ELECTROMAGNETIC STRESS TENSOR

NORMAL FORCE density

$$\mathbf{n}\cdot\underline{\underline{T}}\cdot\hat{n} = \epsilon_0(\mathbf{n}\cdot\mathbf{E})^2 + \mu_0|\mathbf{n}\cdot\mathbf{H}|^2 - \frac{1}{2}\left(\epsilon_0|\mathbf{E}|^2 + \mu_0|\mathbf{H}|^2\right)$$

at the surface of a conductor

$$|\mathbf{n}\cdot\mathbf{E}|^2 = |\mathbf{E}|^2 \qquad |\mathbf{n}\cdot\mathbf{H}|^2 = 0$$

$$\mathbf{n}\cdot\underline{\underline{T}}\cdot\mathbf{n} = \frac{1}{2}\epsilon_0|\mathbf{E}|^2 - \frac{1}{2}\mu_0|\mathbf{H}|^2$$

$$\underset{\sim}{J}_m = \nabla \times \underset{\sim}{M}$$

$$\left.\begin{aligned} \underset{\sim}{D} &= \epsilon_0 \underset{\sim}{E} + \underset{\sim}{P} \\ \underset{\sim}{H} &= \underset{\sim}{B}/\mu_0 - \underset{\sim}{M} \end{aligned}\right\} \text{introduce}$$

$$\boxed{\begin{aligned} \nabla \times \underset{\sim}{E} &= -\frac{\partial B}{\partial t} \\ \nabla \cdot \underset{\sim}{B} &= 0 \\ \nabla \cdot \underset{\sim}{D} &= \rho_f \\ \nabla \times \underset{\sim}{H} &= \left(\underset{\sim}{J}_f + \frac{\partial}{\partial t}\underset{\sim}{D}\right) \end{aligned}}$$

Pointings theorem

$$\frac{dU}{dt} = -\int_V d^3x \; \underset{\sim}{J}_f \cdot \underset{\sim}{E}$$

includes energy required to stretch springs

$$= \int_V d^3x \left[\underset{\sim}{E} \cdot \frac{\partial \underset{\sim}{D}}{\partial t} + \underset{\sim}{H} \cdot \frac{\partial \underset{\sim}{B}}{\partial t}\right]$$

align magnetic dipoles

$$+ \int_S da \; \underset{\sim}{n} \cdot (\underset{\sim}{E} \times \underset{\sim}{H}) \quad 3/\mu_0$$

poynting flux

$$\frac{dU}{dt} = - \int d^3x \left\{ \underset{\sim}{E} \cdot \nabla \times \overset{H}{\cancel{B}} - \underset{\sim}{E} \cdot \frac{\partial}{\partial t} \overset{D}{\cancel{E}} \right\}$$

$$\underset{\sim}{E} \cdot \nabla \times \overset{H}{\cancel{B}} = \nabla \cdot (\overset{H}{\underset{\sim}{B}} \times \underset{\sim}{E}) + \overset{H}{\cancel{B}} \cdot \underbrace{\nabla \times \underset{\sim}{E}}_{-\frac{\partial B}{\partial t}}$$

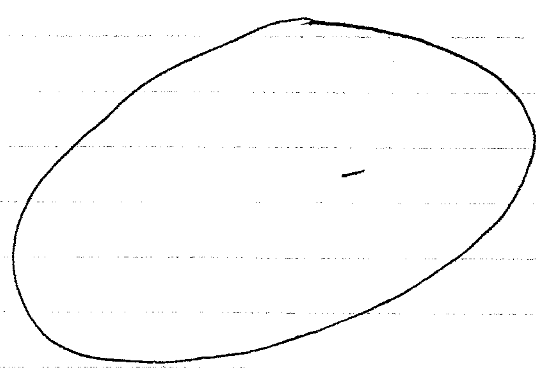
$$= -\nabla \cdot (\underset{\sim}{E} \times \cancel{B})$$

$$\frac{dU}{dt} = + \int d^3x \left[\nabla \cdot (\overset{E \times H}{\cancel{B \times E}}) + \left[\overset{H}{\cancel{B}} \cdot \frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{E} \cdot \frac{\partial \overset{D}{\cancel{E}}}{\partial t} \right] \right]$$

$$\frac{dU}{dt} = \int d^3x \left[\overset{H}{\cancel{B}} \cdot \frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{E} \cdot \frac{\partial \overset{D}{\cancel{E}}}{\partial t} \right]$$

$$+ \int_S da\, \underset{\sim}{n} \cdot (\underset{\sim}{E} \times \overset{H}{\cancel{B}})$$

$\underset{\sim}{B} = \mu \underline{H}$



~~suppose we take S →~~ ∞

Suppose we take $S \to \infty$
and surface term
vanishes

$$\underset{\sim}{E} \cdot \frac{\partial}{\partial t} \underset{\sim}{E} = \frac{\partial}{\partial t} \frac{E^2}{2}$$

THEN

$$\frac{\partial U}{\partial t} = \int d^3x \left[\frac{\cancel{B} H \cdot \frac{\partial B}{\partial t}}{\cancel{4\pi}} + \frac{\underset{\sim}{E} \cdot \frac{\partial \cancel{E} D}{\partial t}}{\cancel{4\pi}} \right]$$

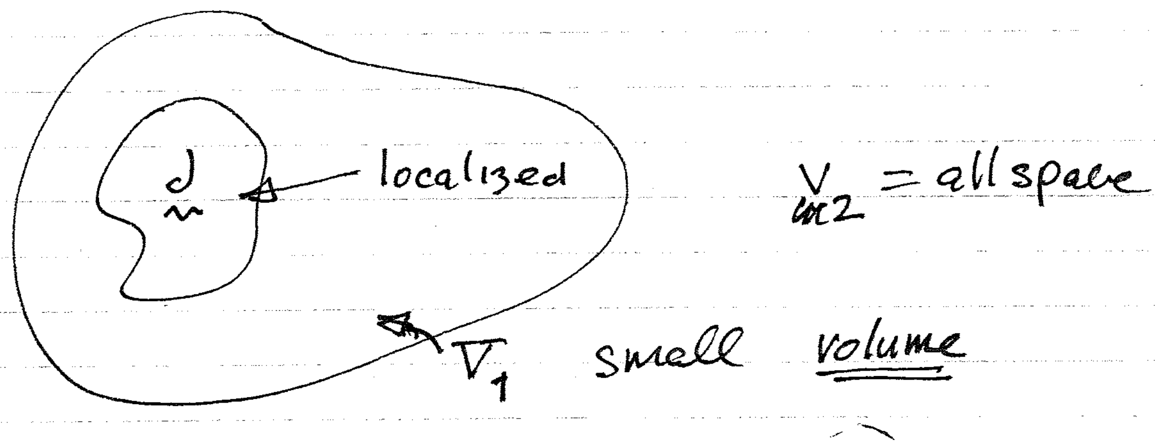
simple case

$$\boxed{\underline{B} = \mu \underline{H} \qquad \underset{\sim}{D} = \epsilon \underline{E}}$$

$$\frac{\partial U}{\partial t} = \frac{d}{dt} \int d^3x \left(\frac{\mu H^2}{\cancel{8\pi}} + \frac{\epsilon E^2}{\cancel{8\pi}} \right) / 2$$

$$\boxed{U = \int_V d^3x \left(\frac{\mu H^2}{\cancel{8\pi}} + \frac{\epsilon E^2}{\cancel{8\pi}} \right) 2}$$

what is the interpretation of the surface term?



what is the rate at which I must supply energy to achieve this?

$$\cancel{\int \underset{\sim}{J} \cdot \underset{\sim}{E} = \int \sum_i q_i}$$

$$\int_V d^3x \, \underset{\sim}{J} \cdot \underset{\sim}{E} = \sum_i \int q_i \underset{\sim}{V}_i \cdot \underset{\sim}{E}(\underset{\sim}{x}_i) = \sum_i \underset{\sim}{V}_i \cdot \underset{\sim}{F}_i$$

THIS IS THE RATE AT WHICH THE ELECTRIC FORCE DOES WORK ON THE SYSTEM OF CHARGES. Rate at which Energy ~~stored~~ transfered to ~~fields~~ fields is opposite

$$\frac{dU}{dt} = - \int d^3x \, \underset{\sim}{J} \cdot \underset{\sim}{E}$$

$$\underset{\sim}{J}_{free} = \left[\nabla \times \underline{H} - \frac{1}{c} \frac{\partial \underset{\sim}{D}}{\partial t} \right]$$

Ampere's LAW

Interpretation

$$\frac{dU}{dt} = \underbrace{\int_{V_1} d^3x \, \frac{\partial}{\partial t} \frac{\epsilon_0 |\underline{E}|^2 + \mu_0 |\underline{H}|^2}{2}}_{} + \underbrace{\int_{S_1} dA \; \hat{\underline{n}} \cdot \underline{E} \times \underline{H}}_{}$$

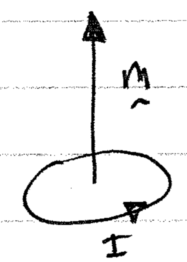
~~Rate of~~ power flowing through surface

$$= \frac{\partial}{\partial t} \left(\begin{matrix} \text{energy} \\ \text{stored in fields} \end{matrix} \right)$$

$\frac{dU}{dt}$ = power delivered to fields in ~~volume 1~~

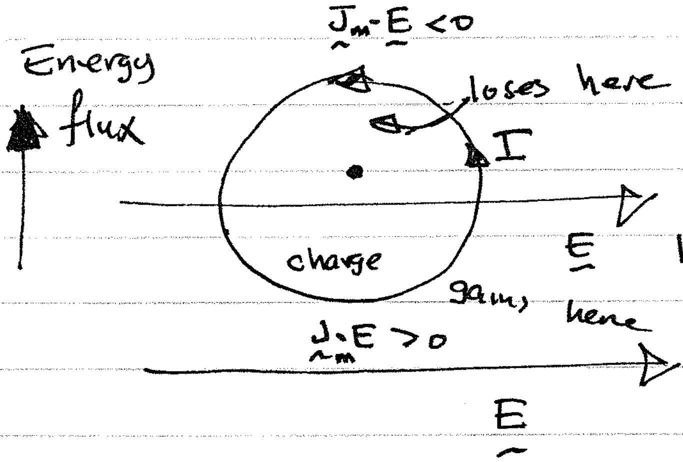
$$\frac{d}{dt} \int d^3x \, \frac{\epsilon_0 |\underline{E}|^2 + \mu_0 |\underline{H}|^2}{2}$$

Rate of increase in field energy



elementary magnetic ~~monopole~~ dipole

viewed from above



$\underset{\sim}{E} \times \underset{\sim}{M}$ down

$$P_m = - c \, \underset{\approx}{E} \times \underset{\approx}{M}$$

Rate of change of momentum

$$\frac{d}{dt} P_{mech} = \int d^3x \left[\rho \underset{\sim}{E} + \underset{\sim}{J} \times \underset{\sim}{B} \right]$$

$$\rho = \nabla \cdot \underset{\sim}{D} \qquad \underset{\sim}{J} = \left(\nabla \times \underset{\sim}{H} - \frac{\partial}{\partial t} \underset{\sim}{D} \right)$$

$$\frac{d}{dt} P_{mech} = \int d^3x \left[\underset{\sim}{E}\, \nabla \cdot \underset{\sim}{D} + \left(\nabla \times \underline{H} - \frac{\partial}{\partial t} \underline{D} \right) \times \underset{\sim}{B} \right]$$

$$\frac{d}{dt} P_{mech} + \int d^3x \frac{\partial}{\partial t} \left(\underset{\sim}{D} \times \underset{\sim}{B} \right) = \int d^3x \left[\underset{\sim}{E}\, \nabla \cdot \underset{\sim}{D} + (\nabla \times \underline{H}) \times \underline{B} + \underset{\sim}{D} \times \frac{\partial \underline{B}}{\partial t} \right]$$

$$= \int d^3x \left[\underline{E}\, \nabla \cdot \underset{\sim}{D} - \underset{\sim}{D} \times \nabla \times \underline{E} + (\nabla \times \underline{H}) \times \underline{B} \right]$$

<u>assume vacuum relations</u>

$$\underset{\sim}{D} = \epsilon_0 \underline{E} \quad + \quad \underset{\sim}{B} = \mu_0 \underline{H}$$

$$\underset{\sim}{E}\, \nabla \cdot \underset{\sim}{E} - \underline{E} \times \nabla \times \underline{E} = \underset{\sim}{E}\, \nabla \cdot \underset{\sim}{E} + \underline{E} \cdot \nabla \underline{E} - \nabla \tfrac{1}{2} |E|^2$$

$$- \underline{H} \times \nabla \times \underline{H} = \underline{H}\, \nabla \cdot \underline{H} + \underline{H} \cdot \nabla \underline{H} - \nabla \tfrac{1}{2} |H|^2$$

$$\frac{d}{dt} \Big(P_{mech} + \int d^3x \underbrace{\underset{\sim}{D} \times \underset{\sim}{B}}_{\epsilon_0 \underline{E} \times \mu \underline{H}} \Big) = \int d^3x\, \nabla \cdot \left[\epsilon_0 \underline{E}\underline{E} + \underline{H}\underline{H} \mu_0 - \underline{\underline{1}}\, \tfrac{1}{2} \left(\epsilon_0 |E|^2 + \mu_0 |H|^2 \right) \right]$$

Other gauge choice $\nabla \cdot \underline{A} = 0$ called the <u>coulomb</u> gauge

Coulomb gauge not as useful.

<u>Green's Function for the Wave Equation in ∞ Free Space</u>

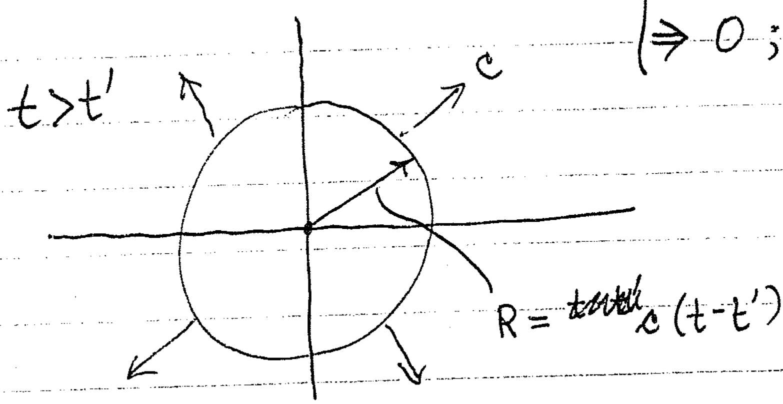
$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -4\pi f(\underline{x}, t)$$

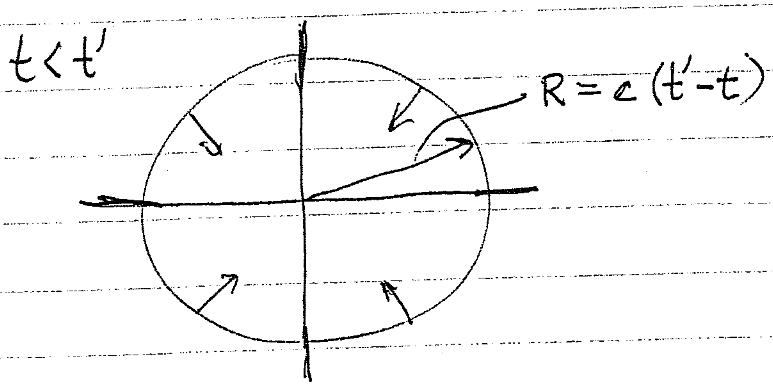
$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G^{\pm}(\underline{x}, t; \underline{x}', t') = -4\pi\, \delta(\underline{x}-\underline{x}')\,\delta(t-t')$$

→ **3D** Two solutions

$$G^{+} = \begin{cases} \delta\left[(t-t') - R/c\right] / R \\ \Rightarrow 0;\ t<t' \quad \neq 0;\ t>t' \end{cases}$$

$$R = |\underline{x} - \underline{x}'|$$





$$G^{-} = \begin{cases} \delta\left[(t-t') + R/c\right] / R \\ \Rightarrow 0;\ t>t' \\ \neq 0;\ t<t' \end{cases}$$

Physics 606 — Chapter 6

Rd. Sections 6.1–5, 6.7–9

Scalar and Vector Potentials

$$\nabla\cdot\underline{B}=0 \quad\Rightarrow\quad \boxed{\underline{B}=\nabla\times\underline{A}}$$

$$\nabla\times E = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}\nabla\times\underline{A}$$

$$\nabla\times\left(E+\frac{\partial \underline{A}}{\partial t}\right)=0$$

$$\Rightarrow \underline{E}+\partial\underline{A}/\partial t = -\nabla\Phi$$

$$\boxed{\underline{E} = -\left(\frac{\partial \underline{A}}{\partial t} + \nabla\Phi\right)}$$

Putting these into (vacuum $\epsilon=\epsilon_0$, $\mu=\mu_0$)

$$\nabla\cdot\underline{E} = \rho/\epsilon_0$$

and $\nabla\times\underline{B} = \mu_0\underline{J} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t}$ $\qquad \frac{1}{c^2}=\mu_0\epsilon_0$

gives

$$\boxed{\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla\cdot\underline{A}) = -\rho/\epsilon_0}$$

$$\underbrace{\nabla\times(\nabla\times\underline{A})}_{\nabla(\nabla\cdot\underline{A})-\nabla^2 A} = \mu_0\underline{J} - \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial\underline{A}}{\partial t}+\nabla\Phi\right)$$

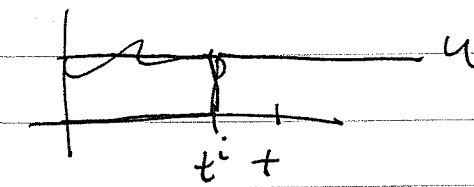
$$\boxed{\nabla^2\underline{A} - \frac{1}{c^2}\frac{\partial^2\underline{A}}{\partial t^2} - \nabla\left(\nabla\cdot\underline{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) = -\mu_0\underline{J}}$$

NOTE: Eqs. for Φ and $\underline{A}$ are coupled. looks nasty

Change problem to unit step function

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_u = -4\pi\,\delta(\underline{x}-\underline{x}')\,u(t-t')$$

$$G = \frac{\partial}{\partial t} G_u$$



$$\frac{\partial u(t-t')}{\partial t} = \delta(t-t')$$

$$\frac{1}{R^2}\frac{\partial}{\partial R} R^2 \frac{\partial}{\partial R} G_u - \frac{1}{c^2}\frac{\partial^2 G_u}{\partial \tau^2} = -\frac{\delta(R)}{R^2}\, u(\tau)$$

$$G_u = \frac{g}{R}$$

$$\frac{\partial^2}{\partial R^2} g - \frac{1}{c^2}\frac{\partial^2 g}{\partial \tau^2} = -\frac{\delta(R)}{R}\, u(\tau)$$

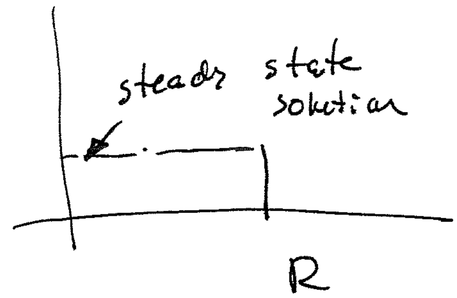
$= 0$ except at $R=0$

for $R>0$ $\qquad g = g_{out}(R-c\tau) + g_{in}(R+c\tau)$

no incoming wave solution

as $\tau > \infty$

$$\frac{\partial^2}{\partial R^2} g - \frac{1}{c^2}\frac{\partial^2 g}{\partial \tau^2} = -\frac{\delta(R)}{R}$$



steady state solution

$$\cancel{g = 1} \qquad g_{out} = \begin{cases} 1 & R - c\tau < 0 \\ 0 & R - c\tau > 0 \end{cases}$$

$$G_u = \frac{1}{R} u(\overset{c\tau - R}{R\ \cancel{\ldots}}) = \frac{1}{R} u(\tau - R/c)$$

$$\boxed{\frac{\partial}{\partial \tau} G_u = \frac{1}{R}\,\delta\!\left(\tau - \frac{R}{c}\right) \neq G}$$

Generally we will be interested in the retarded solution.

Want Ψ to satisfy ~~a radiation~~ an outward radiation condition at ∞.

$\Rightarrow$ Say ① $f(\underline{x},t)$ is localized in space

~~and~~ ② $f=0$ for $t<-T$

~~and~~ ③ $\Psi=0$ for $t<-T$

Then

$$\Psi(\underline{x},t) = \int f(\underline{x}',t')\, G^{(+)}(\underline{x},t;\underline{x}',t')\, \cancel{d^3x'}\, dt'$$

Do t' $\int$:

$$= \int \underbrace{f\left(\underline{x}', t-\frac{|\underline{x}-\underline{x}'|}{c}\right)}_{[f(\underline{x}',t')]_{ret}} \Big/ |\underline{x}-\underline{x}'| \; d^3x'$$

$$= \int \frac{[f(\underline{x}',t')]_{ret}}{|\underline{x}-\underline{x}'|} d^3x'$$

Noting that each of the cartesian components of $\nabla^2 \underline{A} - \frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} = -\mu_0 \underline{J}$ is ~~just the~~ a scalar wave eq. we have

$$\Phi(\underline{x},t) = \frac{1}{4\pi\epsilon_0}\int \frac{[\rho(\underline{x}',t')]_{ret}}{R} d^3x'$$

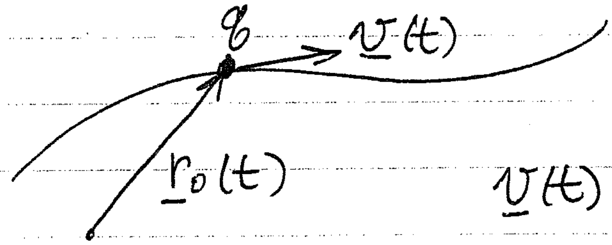
$$\underline{A}(\underline{x},t) = \frac{\mu_0}{4\pi}\int \frac{[\underline{J}(\underline{x}',t')]_{ret}}{R} d^3x'$$

$$t' = t - R/c$$

Field from a moving point charge located at $\underline{x} = \underline{r}_0(t)$

$$\rho(\underline{x}, t) = q\, \delta(\underline{x} - \underline{r}_0(t))$$

$$\underline{J} = q\, \underline{v}(t)\, \delta(\underline{x} - \underline{r}_0(t))$$



$$\underline{v}(t) = \frac{d\underline{r}_0(t)}{dt}$$

Using the retarded Green function:

$$\underline{B} = \frac{\mu_0 q}{4\pi} \left\{ \left[\frac{\underline{v} \times \hat{\underline{R}}}{K R^2} \right]_{ret} + \frac{1}{c} \frac{\partial}{\partial t} \left[\frac{\underline{v} \times \hat{\underline{R}}}{K R} \right]_{ret} \right\}$$

+ Similar eq. for $\underline{E}$

~~① $|\underline{x} - \underline{r}_0(t')| = c(t - t')$ solve for t'~~

~~$\underline{E}$ not so easy~~

WHERE $\quad R(t) = |\underline{x} - \underline{r}_0(t)|$

$$\hat{\underline{R}}(t) = \frac{\underline{x} - \underline{r}_0(t)}{|\underline{x} - \underline{r}_0(t)|}$$

$$K = 1 - \frac{\underline{v}(t) \cdot \hat{\underline{R}}(t)}{c}$$

To use this Eq. for $\underline{B}$ we need to evaluate it at the retarded time t' where

$$|\underline{x} - \underline{r}_0(t')| = c\,(t - t')$$

in general, a nonlinear eq. for t'.

$\underline{A}$ and Φ have an arbitrayness:

"Gauge Transformation"
$$\left\{\begin{array}{l} \underline{A} \to \underline{A}' = \underline{A} + \nabla\Lambda \\ \Phi \to \Phi' = \Phi - \dfrac{\partial \Lambda}{\partial t} \end{array}\right\} \Rightarrow \begin{array}{l} \underline{B} \to \underline{B} \\ \underline{E} \to \underline{E} \end{array}$$

This arbitrayness is a reflection that we can choose $\nabla \cdot \underline{A}$ to be whatever we want.

Lorentz gauge:

$$* \quad \boxed{\nabla \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0}$$

With this gauge the ~~wave~~ equations for the potentials become

"vector wave eq." $\to$ $$\nabla^2 \underline{A} - \frac{1}{c^2}\frac{\partial \underline{A}}{\partial t} = -\mu_0 \underline{J}$$

"scalar wave eq." $\to$ $$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial \Phi}{\partial t^2} = -\rho/\epsilon_0$$

(Statics: $\partial/\partial t = 0 \Rightarrow$
$\nabla^2 \underline{A} = -\mu_0 \underline{J}$
$\nabla^2 \Phi = -\rho/\epsilon_0$)

NOTE: Eqs. for $\underline{A}$ & Φ decouple

Say $\underline{A}, \Phi$ is not in the ~~~~ Lorentz gauge

$$\left(\nabla \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) \neq 0$$

Can we do a gauge transformation to make it so?

Let $\underline{A}' = \underline{A} + \nabla\Lambda$

$$\Phi' = \Phi - \frac{\partial\Lambda}{\partial t}$$

Then

$$\nabla\cdot\underline{A}' + \frac{1}{c^2}\frac{\partial\Phi'}{\partial t} = \left(\nabla\cdot\underline{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) + \left(\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}\right)$$

So $\nabla\cdot\underline{A}' + \frac{1}{c^2}\frac{\partial\Phi'}{\partial t} = 0$ if we can choose Λ so that

the rhs $= 0 \Rightarrow$

$$\boxed{\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\left(\nabla\cdot\underline{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)}$$

~~Wave~~ Eqs. ~~for~~ Λ

~~This equation (as we will~~

This equation the inhomogeneous scalar wave equation is solvable for Λ.

$\Rightarrow$ $\underline{A}', \Phi'$ can be choosen to satisfy the

Lorentz gauge.

Note: There are many potentials that satisfy the Lorentz gauge:

From above, if $(\underline{A}, \Phi)$ satisfied the Lorentz gauge, then so does $(\underline{A}', \Phi')$ provided that

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0.$$

Plane wave propagation

Maxwell's equations with no free sources

$$\nabla \cdot \underset{\sim}{B} = 0$$

$$\nabla \cdot \underset{\sim}{D} = 0$$

$$\nabla \times \underset{\sim}{H} = \cancel{\tfrac{1}{c}}\frac{\partial}{\partial t}\underset{\sim}{D} \qquad \underset{\sim}{D} = \epsilon \underset{\sim}{E}$$

$$\nabla \times \underset{\sim}{E} = -\cancel{\tfrac{1}{c}}\frac{\partial \underset{\sim}{B}}{\partial t} \qquad \underset{\sim}{B} = \mu \underset{\sim}{H}$$

constants

THIS Represents a system of coupled, linear partial differential equations with constant coefficients.

THAT IS c, ϵ, μ are constants

Therefore it is appropriate to look for solutions varying exponentially with the independent variables.

$$\mathbf{B}(\mathbf{x},t) = \frac{1}{2}\Big[\underbrace{\hat{\mathbf{B}}\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + c.c.}_{\text{this is required so that } \mathbf{B} \text{ is real}} \Big]$$

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{2}\Big[\hat{\mathbf{E}}\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + c.c. \Big]$$

$$\mathbf{D}(\mathbf{x},t) =$$

$$\mathbf{H}(\mathbf{x},t) =$$

Hatted quantitities $\quad \nabla \Rightarrow i\mathbf{k}$

$$\frac{\partial}{\partial t} \Rightarrow -i\omega$$

$$\left.\begin{aligned} \mathbf{k}\cdot\hat{\mathbf{B}} &= 0 \\ \mathbf{k}\cdot\hat{\mathbf{D}} &= 0 \end{aligned}\right\} \text{these are un necessary} \qquad \begin{aligned} \hat{\mathbf{D}} &= \epsilon\hat{\mathbf{E}} \\ \hat{\mathbf{B}} &= \mu\hat{\mathbf{H}} \end{aligned}$$

$$i\mathbf{k}\times\hat{\mathbf{H}} = -\frac{i\omega}{\cancel{c}}\hat{\mathbf{D}}$$

$$i\mathbf{k}\times\hat{\mathbf{E}} = \frac{i\omega}{\cancel{c}}\hat{\mathbf{B}}$$

$$i\mathbf{k}\times\overset{\mu H}{\mathbf{B}} = -\frac{i\omega}{\cancel{c}}\epsilon\mu\mathbf{E}$$

$$\underset{\sim}{k} \times \hat{\underset{\sim}{E}} = \frac{\omega}{\cancel{c}} \hat{\underset{\sim}{B}}$$

$$\underset{\sim}{k} \times \hat{\underset{\sim}{B}} = -\frac{\omega}{\cancel{c}} \epsilon\mu \hat{\underset{\sim}{E}}$$

$$\underset{\sim}{k} \times \frac{\omega}{\cancel{c}} \hat{\underset{\sim}{B}} = \underset{\sim}{k} \times (\underset{\sim}{k} \times \hat{\underset{\sim}{E}}) = -\frac{\omega^2}{c^2} \epsilon\mu \hat{\underset{\sim}{E}}$$

$$\underset{\sim}{k} (\underset{\sim}{k} \cdot \hat{\underset{\sim}{E}}) - k^2 \hat{\underset{\sim}{E}} = -\frac{\omega^2}{\cancel{c^2}} \epsilon\mu \hat{\underset{\sim}{E}}$$

follows from $i\underline{k} \times \underline{H} = \underline{D}$ $-i\omega$

$\underline{D} = \epsilon \underline{E}$

note $\underset{\sim}{k} \cdot \underset{\sim}{E} = 0$ thus we have but

if $\epsilon \to \underline{\underline{\epsilon}}$

$\underline{k} \cdot \underline{D} = 0$

$$\left[\frac{\omega^2}{\cancel{c^2}} \epsilon\mu - k^2\right] \hat{\underset{\sim}{E}} = 0 \quad \text{and} \quad \hat{\underset{\sim}{E}} \cdot \underset{\sim}{k} = 0$$

does not imply

$\underline{k} \cdot \underline{E} = 0$

$\underline{k} \cdot \underline{B} = 0$

$\underline{B} = \underline{\underline{\mu}} \cdot \underline{H}$

$\underline{k} \cdot \underline{H} \neq 0$

<u>ABOVE</u> can only be satisfied

if

$$\omega^2 = k^2 \cancel{c^2} / \epsilon\mu = k^2 c_{\phi}^2 \qquad c_{\phi}^2 = \frac{c^2}{\epsilon\mu}$$

~~V_p = phas~~ $\epsilon = \mu = 1$ $\quad c_\phi = c$

$c = \frac{1}{\sqrt{\epsilon\mu}}$

$\omega = \pm ck$ a dispersion relation

for a given k there

$\omega = \omega(\underset{\sim}{k})$ is a unique ω

$\underset{\sim}{E}$ must be transverse to $\underset{\sim}{k}$

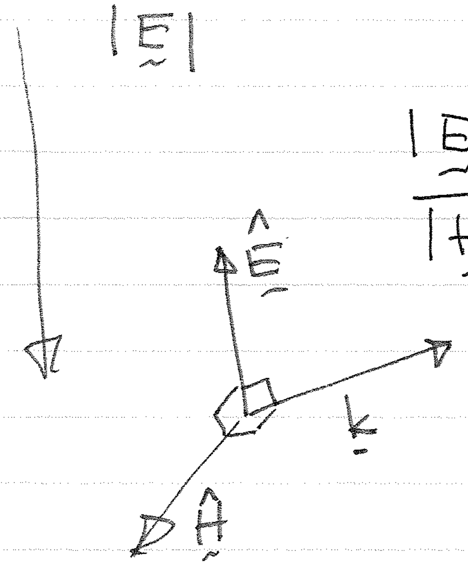
$\frac{\omega}{c} \hat{\underset{\sim}{B}} = \underset{\sim}{k} \times \hat{\underset{\sim}{E}}$ $\hat{B}$ is transverse to both $\underset{\sim}{k}$ and $\underset{\sim}{E}$

$\underset{\sim}{B} = \mu \underset{\sim}{H}$

$$\frac{|\underset{\sim}{H}|}{|\underset{\sim}{E}|} = \frac{|\underset{\sim}{B}|}{\mu |\underset{\sim}{E}|} = \left|\frac{kc}{\omega\mu}\right| = \left|\frac{kc}{\mu \frac{c}{\sqrt{\epsilon\mu}}}\right| = \sqrt{\frac{\epsilon}{\mu}}$$

$$\frac{|\underset{\sim}{E}|}{|\underset{\sim}{H}|} = \frac{\omega\mu}{|\underset{\sim}{k}|} \qquad |\underset{\sim}{k}| = \omega\sqrt{\epsilon\mu} \qquad \frac{|\underset{\sim}{E}|}{|\underset{\sim}{H}|} = \sqrt{\frac{\mu}{\epsilon}} = \eta$$

impedance ~~of free space~~ $377\,\Omega$



for propagation in vacuum $\epsilon = \mu = 1$

$\cancel{|B| = |E|}$