Chapter 23

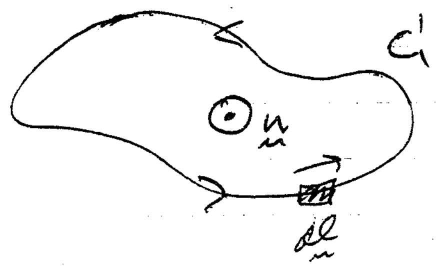
Ch 6

Faradays Law of Induction

Up to this point we have considered magnetic fields ~~produced~~ and electric fields produced by ~~steady~~ ~~static~~ currents and charge distributions. ~~[illegible]~~ We now want to begin to generalize our results to allow for time variation.

Consider a circuit C as shown. The total magnetic flux linking this ~~[illegible]~~ circuit is

changing $\underset{\sim}{B} \to \underset{\sim}{E}$



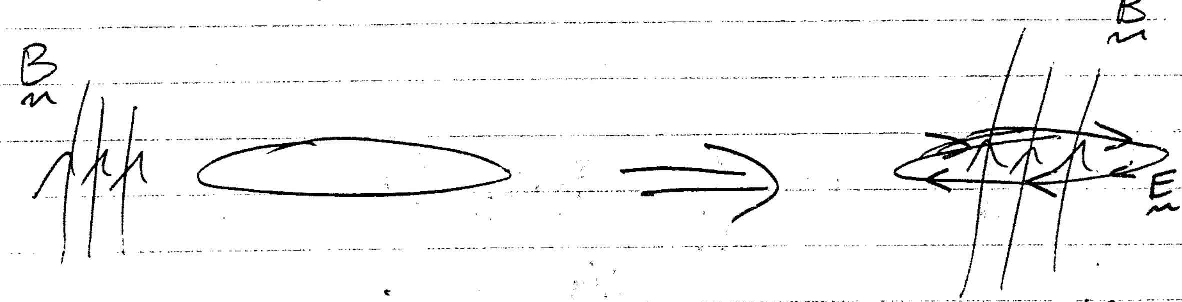
$$\psi = \int_S d\underset{\sim}{s} \cdot \underset{\sim}{B}$$

Observations are that an electric field is produced in the circuit if the magnetic flux F changes in time. Define

$$\mathcal{E} = \oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = \text{electromotive force}$$

$$\boxed{\mathcal{E} = - dF}$$ Faradays

the direction of the electric field is such that the resulting current would oppose the change in the magnetic field (Lenz' Law)



can generalize this result to an arbitrary closed loop in space.

$$\mathcal{E} = \oint_{C} \underset{\sim}{E} \cdot d\underset{\sim}{l} = \oint_{S'} \nabla \times \underset{\sim}{E} \cdot d\underset{\sim}{S}$$

$$\int d\underset{\sim}{s} \cdot \left[\nabla \times \underset{\sim}{E} + \frac{\partial \underset{\sim}{B}}{\partial t} \right] = 0 .$$

Since this is valid for any surface S', the result must be valid locally at every point

$$\boxed{\nabla \times \underset{\sim}{E} + \frac{\partial B}{\partial t} = 0}$$

differential form of Faraday's law.

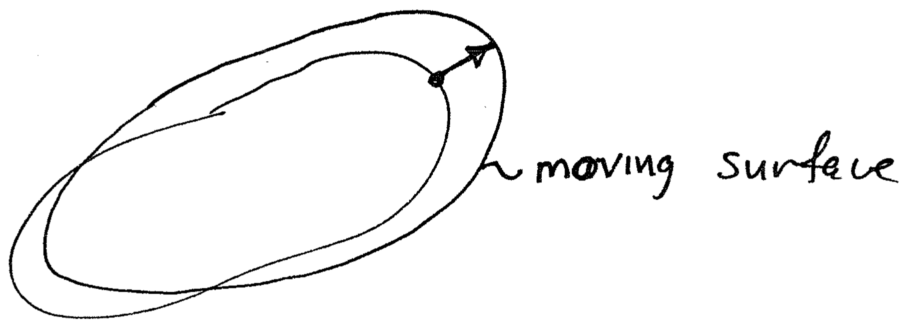
Note that for $\partial B / \partial t = 0$ this reduces to

$\nabla \times E = 0$ as we assumed previously consider a magnetic field

$\underset{\sim}{B}(\underset{\sim}{x}, t)$ and

associated $\underset{\sim}{E}(\underset{\sim}{x}, t)$

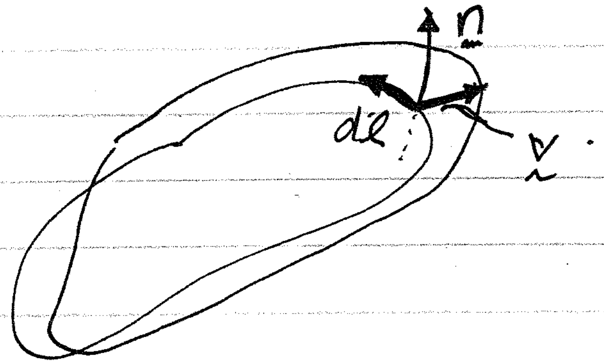
$$\cancel{d} \oint_{L(} \qquad \frac{d}{dt} \overset{F}{\Phi}(t) = \frac{d}{dt} \int_{S(t)} (\underset{\sim}{B} \cdot \underset{\sim}{n}) da$$



$$\frac{d}{dt} \int_{S(t)} (\underset{\sim}{B} \cdot \underset{\sim}{n}) da = \int_{S(t)} \frac{\partial}{\partial t} \underset{\sim}{B} \cdot n \, da +$$

$$\frac{1}{\delta t} \int_{\delta S(t)} \underset{\sim}{B} \cdot n \, da \, \delta a$$

Let $\underset{\sim}{V}(\underset{\sim}{x},t)$ be the velocity of the element of the surface



$$\underset{\sim}{n}\, d\,\delta a = -d\underset{\sim}{l} \times \underset{\sim}{V}\,\delta t \quad =$$

$$-\,c\,\nabla \times \underset{\sim}{E}$$

$$\frac{d}{dt}\int_{S(t)} (\underset{\sim}{B}\cdot\underset{\sim}{n})\,da = \int_{S(t)} da\, \underset{\sim}{n}\cdot\frac{\partial B}{\partial t}$$

$$-\int_C \underset{\sim}{B}\cdot d\underset{\sim}{l}\times\underset{\sim}{V}$$

$$-\int_C d\underset{\sim}{l}\cdot\underset{\sim}{V}\times\underset{\sim}{B}$$

For moving loop

$$\frac{d}{dt}\Phi = -\int_{C(t)} d\ell \cdot \left[E + \frac{V \times B}{c} \right]$$

$$= -\int_{C(t)} d\ell \cdot E' =$$

$$E' = E + V \times B$$

$$\mathcal{E}MF = \int d\ell \cdot E' = -\frac{d\Phi}{dt}$$

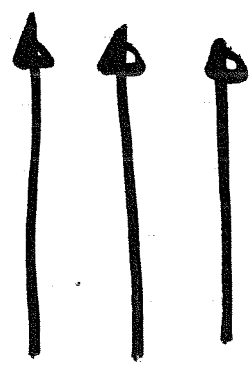
<u>electromotive force</u>

$$E' = E + V \times B$$

"electric field felt
by charge movin with
velocity V"

get examples

Homopolar generator

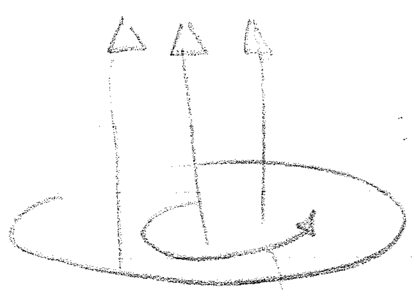


$\underset{\sim}{B} = B_0 \underset{\sim}{e}_z$ constant

$\underset{\sim}{E} = 0$

Now add spining <u>metal</u> disc

spining ~~[illegible]~~ disc there is



Force $\propto$ $\underset{\sim}{E} + \frac{\underset{\sim}{V} \times \underset{\sim}{B}}{c} = \underset{\sim}{E}'$

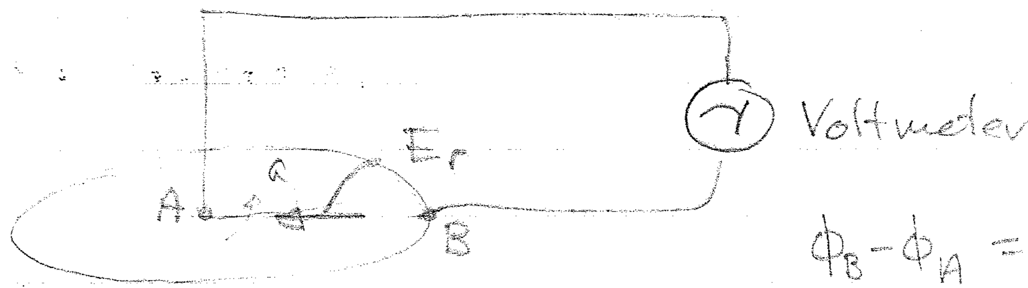
$\underset{\sim}{V} = V_\theta \underset{\sim}{e}_\theta = \Omega r \underset{\sim}{e}_\theta$

For a good conductor $\underset{\sim}{E}' = 0$

$E_r + \frac{V_\theta B_0}{c} = 0 \qquad E_r = -\frac{\Omega r B_0}{c}$

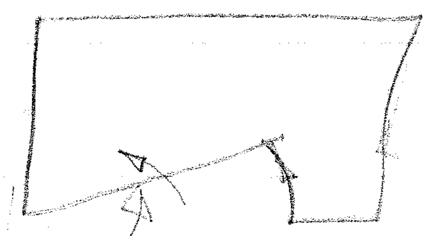
electrostatic field

$\phi = \frac{\Omega r^2}{c} B_0$



$$\phi_B - \phi_A = \frac{\Omega a^2 B_0}{c}$$

$$\mathcal{E}mf = \oint d\ell \left(\underset{\sim}{E} + \frac{\underset{\sim}{V} \times \underset{\sim}{B}}{c} \right)$$



$$\int \int d\underset{\sim}{\ell} \cdot \left(\underset{\sim}{E} + \frac{\underset{\sim}{V} \times \underset{\sim}{B}}{c} \right)$$

around stationary loop = 0

$$\underset{\sim}{E} = -\nabla \phi$$

if $d\hat{\ell}$ element is embedded in nonmoving [illegible] disc

$$\int d\ell \cdot \left(\underset{\sim}{E} + \frac{\underset{\sim}{V} \times \underset{\sim}{B}}{c} \right)$$

$$= \int_0^a dr \frac{V_\theta B_0}{c} = \frac{\Omega a^2 B_0}{c}$$

Dynamo

resistivity $= \frac{1}{\sigma}$

$$\underline{E} + \underset{\sim}{V} \times \underset{\sim}{B} = \eta \underset{\sim}{J}$$

$$\underline{E} = \eta \underset{\sim}{J} - (\underset{\sim}{V} \times \underset{\sim}{B})$$

$$+\frac{\partial}{\partial t}\underset{\sim}{B} = -\nabla \times \underline{E}$$

$$\underset{\sim}{J} = \frac{1}{\mu_0} \nabla \times \underset{\sim}{B}$$

$$= -\nabla \times \eta \frac{1}{\mu_0}(\nabla \times B) + \nabla \times (\underset{\sim}{V} \times \underline{B})$$

$$\frac{\partial}{\partial t}\underset{\sim}{B} = -\frac{\eta}{\mu_0}\left[\nabla(\nabla \cdot \underset{\sim}{B}) - \nabla^2 \underset{\sim}{B}\right] + \underset{\sim}{B} \cdot \nabla \underset{\sim}{V} + \nabla \cdot \underset{\sim}{B}\, \underset{\sim}{V}$$

$$- \underset{\sim}{V} \cdot \nabla \underset{\sim}{B} - \underset{\sim}{B}\, \nabla \cdot \underset{\sim}{V}$$

NOW LET'S $(\nabla \cdot \underset{\sim}{B}) = 0$

ASSUME $\nabla \cdot \underset{\sim}{V} = 0$ incompressible flow

$\underline{V}$ – given

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{V} \cdot \nabla \underset{\sim}{B} = \underset{\sim}{B} \cdot \nabla \underset{\sim}{V} + \left(\frac{\eta}{\mu_0}\right) \nabla^2 \underset{\sim}{B}$$

time derivative following flow

AMPLIFICATION ATTENUATION due to flow inhomogeneity

DIFFUSION of $\underset{\sim}{B}$ due to resistivity (can be very small in a good conductor) what is Behaviour of $\underset{\sim}{B}(\underset{\sim}{x},t)$?

Suppose $\underset{\sim}{V}$ = const

$$\frac{\partial}{\partial t}\underset{\sim}{B} + \underset{\sim}{V}\cdot\nabla\underset{\sim}{B} = \left(\frac{\eta}{\mu_0}\right)\nabla^2\underset{\sim}{B}$$

all coefficients independant of $\underset{\sim}{x}$, t

Look for

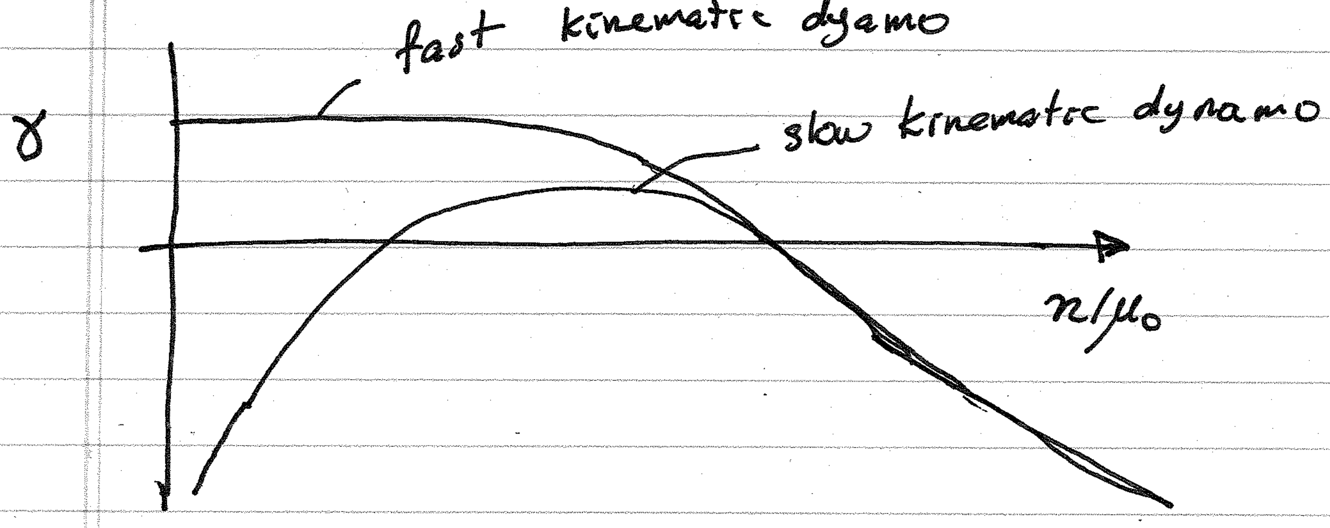
$$\underset{\sim}{B} = \text{Re}\left\{ \hat{\underset{\sim}{B}}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} + \gamma t} \right\}$$

$$(\gamma + i\underset{\sim}{k}\cdot\underset{\sim}{V})\hat{\underset{\sim}{B}} = -\frac{\eta}{\mu_0}k^2\hat{\underset{\sim}{B}}$$

$$\gamma = i\underset{\sim}{k}\cdot\underset{\sim}{V} - \frac{\eta}{\mu_0}k^2 \qquad \text{Re}\{\gamma\} < 0$$

field decays in time!

What if $\underset{\sim}{V}$ is inhomogeneous
Then the following is possible



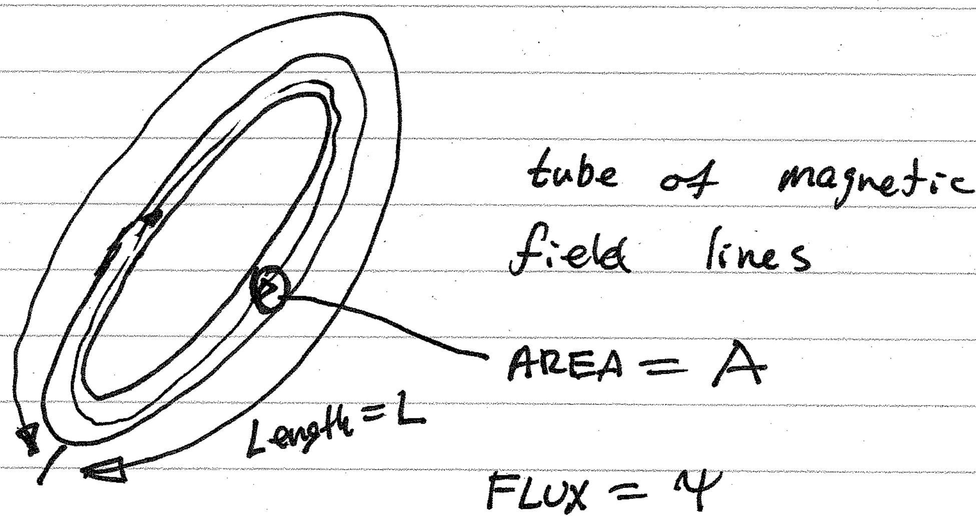
If $\gamma > 0$ small seed $\underset{\sim}{B}$ is amplified in time.

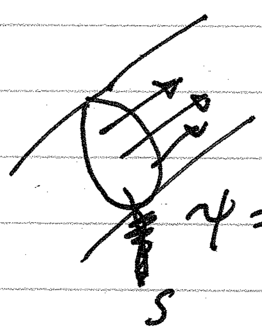
ORIGIN OF: TERRESTRIAL, SOLAR, GALACTIC MAGNETIC FIELDS

WHAT KIND OF $\underset{\sim}{v}(\underline{x},t)$ is needed?

ANS: stretch – twist – fold

CONSIDER CARTOON:

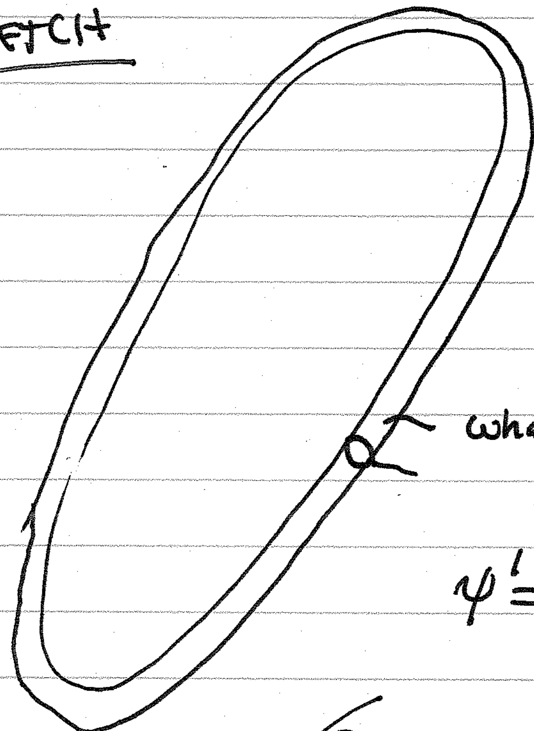




$$\psi = \int_S \underset{\sim}{B} \cdot \underline{n}\, da$$

Imagine flow stretches tube to 2L

STRETCH



$L' = 2L$. why

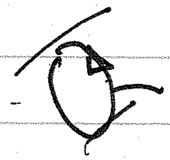
$A' = A/2$ $\quad \nabla \cdot \underset{\sim}{v}$

$V = A'L' = AL$

$\cancel{V} =$ const

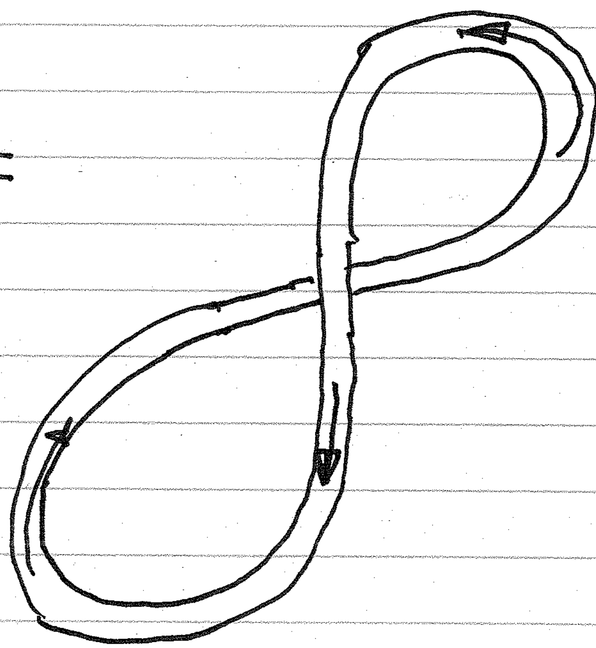
what is ψ'?

$\psi' = \psi$

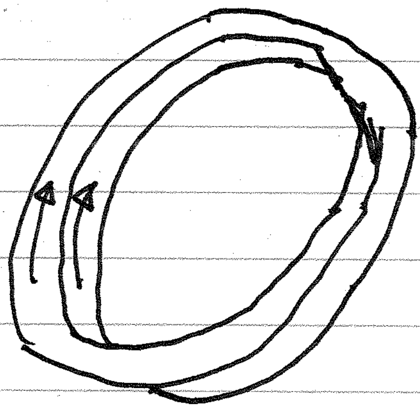


$\int dl \, (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) = -\frac{d\psi}{dt} = 0$

TWIST



FOLD New loop



$$L'' = L'/2 = L$$

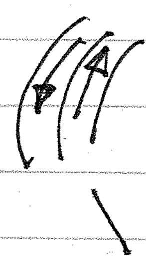
$$A'' = 2A' = A$$

$$\psi'' = 2\psi$$

FIELD HAS ~~DOU~~ DOUBLED !

Picture is actually more complicated

sometimes stretching and twisting results in this situation



resistivity (flux diffusion causes cancellation)