Chapter 22

Magneto Static Boundary Value Problems

$$\nabla \cdot \underset{\sim}{B} = 0$$

$$\nabla \times \underset{\sim}{H} = \underset{\sim}{J}_{free} \qquad \underset{\sim}{B} = \mu \underset{\sim}{H}$$

OR

$$\underset{\sim}{B} = \mu_0 (\underset{\sim}{H} + \underset{\sim}{M})$$

if $\underset{\sim}{M}$ is known

Suppose $\underset{\sim}{J}_{free} = 0$

a) $\nabla \cdot \underset{\sim}{B} = 0 \Rightarrow$ allows us to write $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$

$\nabla \times \underset{\sim}{H} = 0 \Rightarrow$ allows us to write $\underset{\sim}{H} = -\nabla \Phi_m$

Pick $\underset{\sim}{H} = -\nabla \Phi_m$ because solving for a scalar is easier than solving for a vector

$$\nabla \cdot \underset{\sim}{B} = 0 \Rightarrow \nabla \cdot \mu \underline{H} = 0$$

or

$$\nabla \cdot \mu_0 (\underline{H} + \underline{M}) = 0$$

case 1) linear material

$$\nabla \cdot \mu \underset{\sim}{H} = - \nabla \cdot (\mu \nabla \phi_m) = 0$$

in regions of constant μ

$$\nabla^2 \phi_m = 0$$

BCs $\underset{\sim}{H}_{t1} = \underset{\sim}{H}_{t2}$ (no free surface currents)

ϕ_m is continuous

$\underset{\sim}{n} \cdot \underset{\sim}{B} = n \cdot (\mu \underset{\sim}{H}) = - \underset{\sim}{n} \cdot \mu \nabla \phi_m$ continuous

$$\mu_1 \frac{\partial \phi}{\partial x_n} = \mu_2 \frac{\partial \phi}{\partial x_n}$$ Just like electrostatics

ϕ continuous

$$\epsilon_1 \frac{\partial \phi}{\partial x_n} = \epsilon_2 \frac{\partial \phi}{\partial x_n}$$

Continuity of B_n $\quad \mu_1 \frac{\partial \phi_m}{\partial r}\Big|_{\text{inside}} = \mu_0 \frac{\partial \phi_m}{\partial r}\Big|_{\text{outside}}$

$\cancel{\mu H_i} =$

$$\mu_1 \left[-H_i \right] = \mu_0 \left[-\frac{B_0}{\mu_0} - \frac{2m}{4\pi a^3} \right]$$

$$H_i = \frac{\mu_0}{\mu_1} \left[\frac{B_0}{\mu_0} + \frac{2m}{4\pi a^3} \right] = \left[\frac{B_0}{\mu_0} - \frac{m}{4\pi a^3} \right]$$

$$\frac{3m}{4\pi a^3} = \left(\frac{1}{\mu_0} - \frac{1}{\mu_1} \right) B_0$$

$$\boxed{m = \frac{4\pi a^3}{3} \left(\frac{\mu_1 - \mu_0}{\mu_1 \mu_0} \right) B_0}$$

if $\mu_1 > \mu_0$

$\underset{\sim}{m} \parallel \underset{\sim}{B}_0$

$$H_i = \left[1 - \frac{1}{3} \frac{\mu_1 - \mu_0}{\mu_1} \right] \frac{B_0}{\mu_0}$$

~~M inside~~ $= B$

$\underset{\sim}{B}$ inside

$$B_i = \mu_1 H_i = \left(\frac{2}{3}\mu_1 + \frac{1}{3}\mu_0 \right) B_0 / \mu_0$$

Case # 2 Hard ferromagnetic material

$\underset{\sim}{M}$ is specified function of position

$$\nabla \cdot \left(- \nabla \phi_m + \underset{\sim}{M} \right) = 0$$

$$\nabla^2 \phi_m = \nabla \cdot \underset{\sim}{M}$$

$$\phi_m = - \frac{1}{4\pi} \int d^3x' \, \frac{\nabla' \cdot \underset{\sim}{M}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$$\phi_m = \frac{1}{4\pi} \int d^3x' \, \underset{\sim}{M}(x') \cdot \overset{(-\nabla)}{\nabla'} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$$\phi_m = - \frac{1}{4\pi} \int d^3x' \, \underset{\sim}{M}(\underset{\sim}{x}') \cdot \frac{(\underset{\sim}{x}' - \underset{\sim}{x})}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}$$

or

$$\phi_m = - \nabla \cdot \frac{1}{4\pi} \int d^3x \, \frac{\underset{\sim}{M}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

Example permeable sphere in external magnetic field

$$\underset{\sim}{B} = B_0 \hat{z}$$

$$\underset{\sim}{H} = \frac{B_0}{\mu_0} \hat{z}$$

$\mu = \mu_0$

as $r \to \infty$

$\mu = \mu_1$

Outside sphere

$$\nabla^2 \phi_m = 0 \qquad \phi_m \to -\frac{B_0}{\mu_0} r\cos\theta$$

as $r \to \infty$

$$\phi_m = \cos\theta \Big(\overbrace{-\frac{B_0}{\mu_0} r + \frac{m}{4\pi r^2}}^{\text{Two constants}} \Big)$$

m = dipole moment

Inside sphere

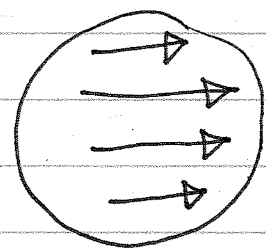
H inside sphere

$$\phi_m = -\cos\theta \, H_i \, r$$

continuity of ϕ

$$H_i = \frac{B_0}{\mu_0} - \frac{m}{4\pi a^3}$$

Problem # 2 Hard Ferromagnetic mat



magnetized sphere with MAGNETIZATION density $\underset{\sim}{M}_0 = \text{const} = M_0 \hat{z}$

$\nabla \times \underset{\sim}{H} = 0 \qquad H = -\nabla \phi_m$

$\cancel{\nabla \cdot (H}$

$\nabla \cdot \underset{\sim}{B} = \nabla \cdot \mu_0 \left(-\nabla \phi_m + \underset{\sim}{M}_0 \right)$

inside $\nabla \cdot \underset{\sim}{M}_0 = 0$

outside $\nabla \cdot \underset{\sim}{M}_0 = 0$

outside:

$$\phi_m = \frac{\cos\theta \, m}{4\pi r^2}$$

inside $\phi_m = -r H_i \cos\theta$

Continuity of ϕ_m

$$\boxed{\frac{m}{4\pi a^2} = -a H_i}$$

Boundary condition

$$\hat{r}\cdot\left(-\nabla\phi_m\right)\Big|_{\text{outside}} = \hat{r}\left(-\nabla\phi_m + \hat{z}M_0\right)\Big|_{\text{inside}}$$

$$+\frac{2m}{4\pi a^3}\cancel{\cos\theta} = H_i \cancel{\cos\theta} + \cancel{\cos\theta}\, M_0$$

$$\frac{2m}{4\pi a^3} = -\frac{m}{4\pi a^3} + M_0$$

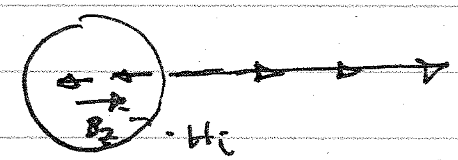
$$m = \frac{1}{3}4\pi a^3 M_0 \qquad \text{(Not surprising)}$$

$$H_i = -\frac{m}{4\pi a^2}$$

outside $\underset{\sim}{H} = -\nabla \phi_m \qquad \phi_m = \dfrac{\cos\theta\, m}{4\pi r^2}$

$$m = \frac{4}{3}\pi a^3 M_0$$

$\cancel{H_z =} \qquad H_z =$



$H_z = \dfrac{2}{4\pi r^2} m$ along axis

inside $\quad H_i = -\dfrac{m}{4\pi a^2}$

$\underset{\sim}{B} = \mu_0 (\underset{\sim}{H} + \underset{\sim}{M})$ $\qquad$ outside $B_z = \dfrac{\mu_0}{2\pi r^3} m$

inside $\cancel{B_z}$

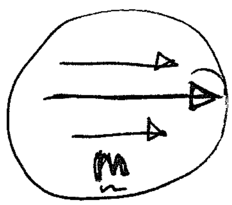
$$B_z = \mu_0 \left(-\frac{m}{4\pi a^2} + M_0 \right)$$

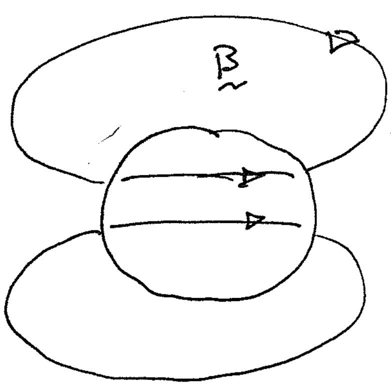
$$\frac{3}{4\pi a^2} m$$

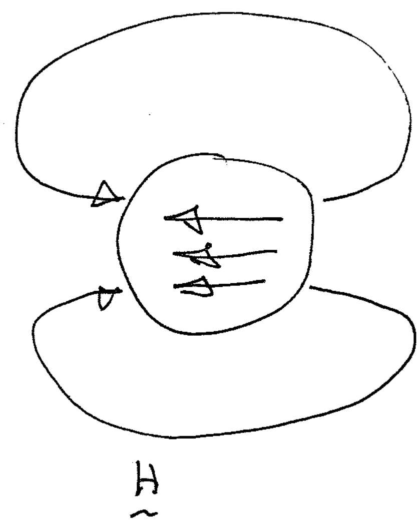
~~∴~~ B_z is continuous

$$\underset{\sim}{H}_i = \hat{z} H_i = \hat{z}\left(-\frac{m}{4\pi a^3}\right) = \hat{z}\left(-\frac{1}{3} M_0\right)$$

$$\underset{\sim}{B}_i = \mu_0 (H_i + M_0) = \frac{2}{3}\mu_0 M_0$$







add uniform B_0

$\underset{\sim}{M} \longrightarrow M_0$

$\underset{\sim}{H} \longleftarrow -\frac{1}{3} M_0$

$B \longrightarrow \frac{2}{3}\mu_0 M_0$

$$\underset{\sim}{M} = M_0$$

$$\underset{\sim}{H} = -\frac{1}{3} M_0 + \frac{B_0}{\mu_0}$$

$$\underset{\sim}{B} = \frac{2}{3}\mu_0 M_0 + B_0$$

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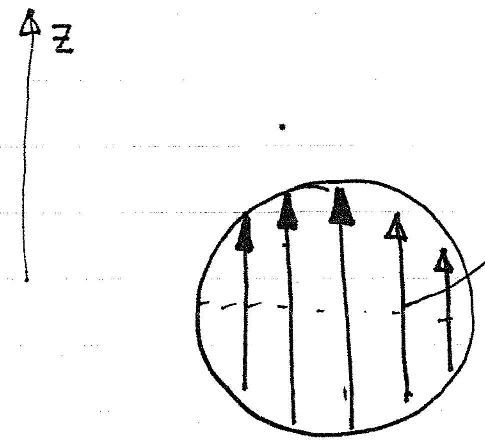
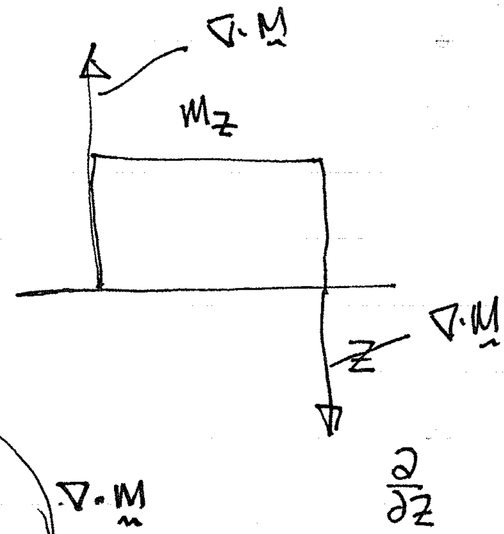


$$\underset{\sim}{M} = \underset{\sim}{e}_z M_o$$



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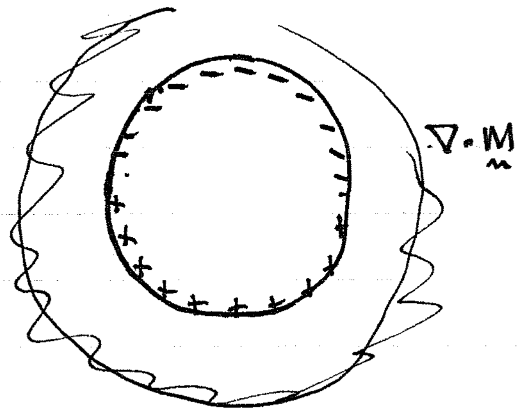
$$\nabla \cdot \underset{\sim}{M} = \frac{\partial}{\partial z} M_z$$



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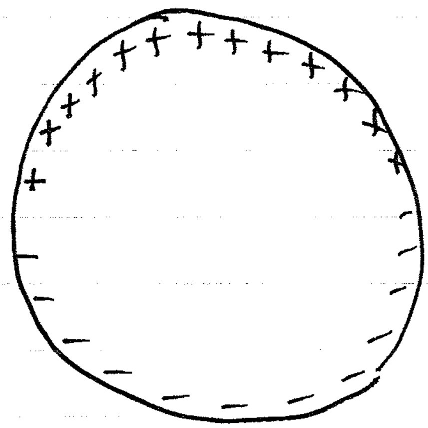
$$\rho_m = -\nabla \cdot \underset{\sim}{M}$$



$\underset{\sim}{H}$

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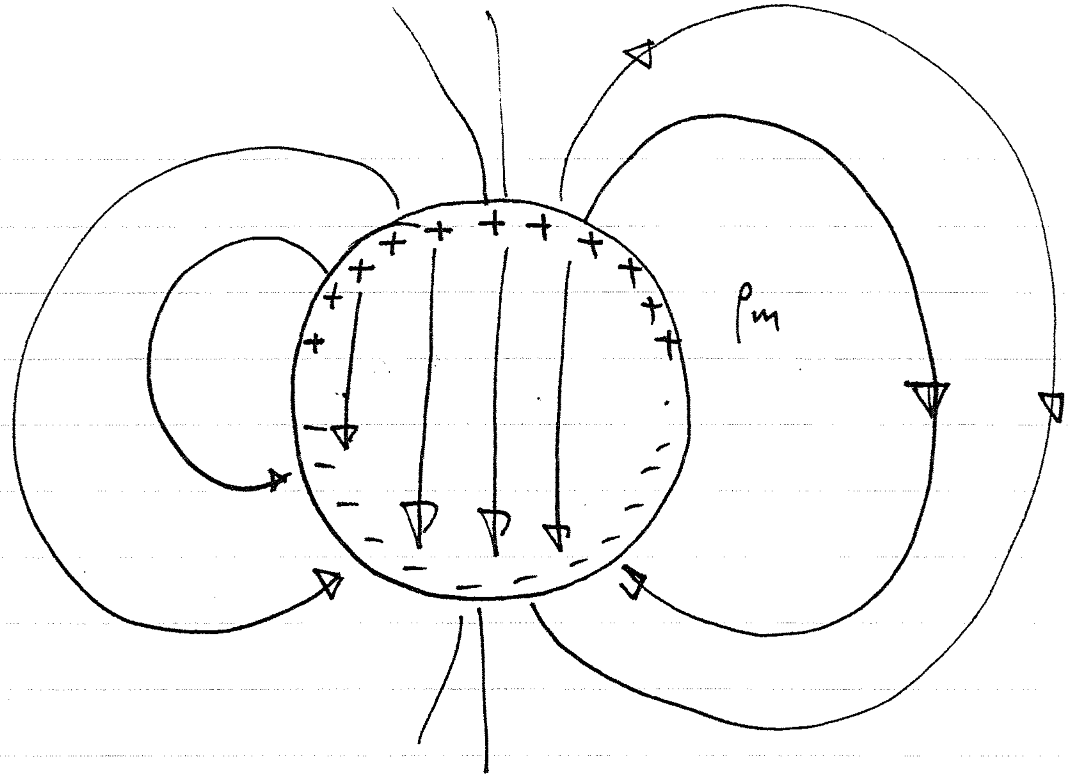

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$\underset{\sim}{B} = \underset{\sim}{H} + 4\pi \underset{\sim}{M}$