Chapter 21

Potential Energy of a <u>permanent</u> dipole $\underset{\sim}{m}$ in an external field

$$U = - \underset{\sim}{m} \cdot \underset{\sim}{B}_{ext}$$

in an external field $\nabla \times \underset{\sim}{B}_{ext} = 0$

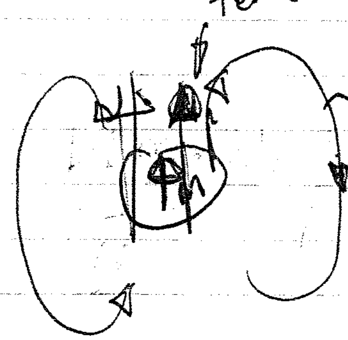
Can not derive our result yet since time dependence of $\underset{\sim}{J}$ not considered.

Lowest Energy State

$\underset{\sim}{m} \parallel \underset{\sim}{B}_{ext}$

tend to allign

$\sim \underset{\sim}{B}$ due to $\underset{\sim}{m}$



To Here end of 10th lecture

<u>12 pages</u>

Review

$$\nabla \times \underset{\sim}{B} = \frac{4\pi}{c} \underset{\sim}{J} \qquad \nabla \cdot \underset{\sim}{B} = 0$$

Force on a current distribution

$$\underset{\sim}{F} = \int d^3x \, \frac{\underset{\sim}{J}(\underset{\sim}{x}) \times \underset{\sim}{B}(\underset{\sim}{x})}{c}$$

Vector potential

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$

Coulomb Guage $\quad \nabla \cdot \underset{\sim}{A} = 0$

$$\underset{\sim}{A} = \frac{1}{c} \int d^3x' \, \frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

Vector Potential due to a localized current distribution $(|\underset{\sim}{x}| \gg |\underset{\sim}{x}'|)$

$$\underset{\sim}{A} = -\frac{\mu_0/4\pi}{|\underset{\sim}{x}|^3}\,\underset{\sim}{x} \times \underset{\sim}{m} \quad = \frac{\underset{\sim}{m} \times \underset{\sim}{x}}{|\underset{\sim}{x}|^3}$$

magnetic moment

$$\underset{\sim}{m} = \frac{1}{2}\int d^3x'\, \underbrace{\underset{\sim}{x}' \times \underset{\sim}{J}(\underset{\sim}{x}')/2}_{\underset{\sim}{M}(\underset{\sim}{x}')}$$

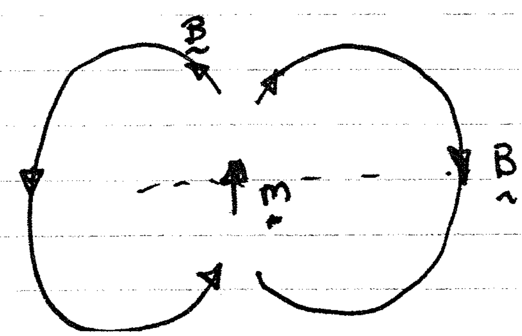
$\underset{\sim}{M}(\underset{\sim}{x}')$ magnetic moment density

<u>magnetization</u>

outside region of current

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} = \frac{3\hat{\underset{\sim}{x}}\,\underset{\sim}{m}\cdot\hat{\underset{\sim}{x}} - \underset{\sim}{m}}{|\underset{\sim}{x}|^3} = -\nabla\left(\frac{\underset{\sim}{M}\cdot\underset{\sim}{x}}{x^3}\right)$$

<u>magnetic dipole field</u>



Relation between magnetic moment and angular momentum

$$\underset{\sim}{m} = \frac{1}{2\cancel{c}} \int \underset{\sim}{x}' \times \underset{\sim}{J}(\underset{\sim}{x}')\, d^3\underset{\sim}{x}'$$

suppose $\underset{\sim}{J}(x')$ is composed of an ensemble of charges with charges q and mass m. each charge has an instantaneous position $\underset{\sim}{x}_i$ and velocity $\underset{\sim}{v}_i$

delta function

$$\underset{\sim}{J}(\underset{\sim}{x}') = q \sum_{i=1}^{N} \underset{\sim}{v}_i \underbrace{\delta(\underset{\sim}{x}' - \underset{\sim}{x}_i)}_{\text{density}}$$

why?

Remember for continuous

$\int_V d^3x\, \rho = $ # of charges in V

$$\underset{\sim}{J} = \rho \underset{\sim}{v} \qquad \rho \simeq \sum_i \delta(\underset{\sim}{x}' - \underset{\sim}{x}_i)$$

$$\underset{\sim}{m} = \frac{q}{2\cancel{c}m} \sum_i \underset{\sim}{x}_i \times (m\underset{\sim}{v}_i) \qquad \underset{\sim}{L}_i = m\underset{\sim}{x}_i \times \underset{\sim}{v}_i$$

$\mathbf{L}_i$ = angular momentum of ith charge.

total angular mom vector

$$\mathbf{m} = \frac{e}{2m\cancel{c}} \sum_i \mathbf{L}_i = \frac{e}{2m\cancel{c}} \mathbf{L}$$

Classical Result

Forces and Torques on localized current distribution

$$\mathbf{F} = \int d^3\mathbf{x}' \, \underbrace{\frac{\mathbf{J}(\mathbf{x}') \times \mathbf{B}(\mathbf{x}')}{\cancel{c}}}_{\text{Force density}}$$

suppose ~~$\mathbf{B}(\mathbf{x}')$~~ $\mathbf{J}(\mathbf{x}')$ is localized to a region of size R and $\mathbf{B}(\mathbf{x}')$ varies smoothly over this region.

Taylor Expand $\underset{\sim}{B}$

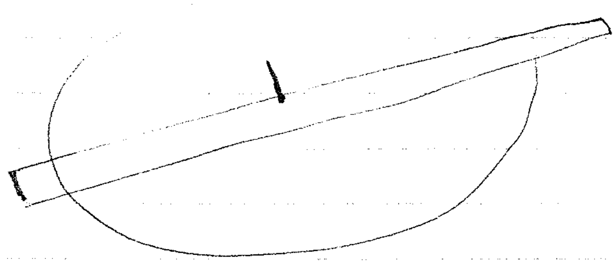
$$\underset{\sim}{B}(\underset{\sim}{x}) = \underset{\sim}{B}(0) + \underset{\sim}{x}\cdot\left(\nabla \underset{\sim}{B}(\underset{\sim}{x})\right)_{\underset{\sim}{x}=0} + \cdots$$

$$\underset{\sim}{F} \simeq \frac{1}{c}\int d^3\underset{\sim}{x}' \; \underset{\sim}{J}(\underset{\sim}{x}') \times \underset{\sim}{B}(0)$$

$$+ \frac{1}{c}\int d^3x' \; \underset{\sim}{J}(\underset{\sim}{x}') \times \left(\underset{\sim}{x}'\cdot\nabla \underset{\sim}{B}(\underset{\sim}{x})\right)_{\underset{\sim}{x}=0} + \cdots$$

FIRST TERM vanishes $\int d^3\underset{\sim}{x}' \, \underset{\sim}{J}(\underset{\sim}{x}) = 0$

$(\nabla\cdot\underset{\sim}{J}) = 0$



~~Second term~~ $\cancel{\frac{1}{c}\int d^3x' \, \underset{\sim}{J}(\underset{\sim}{x}') \times (\underset{\sim}{x}'\cdot\nabla \underset{\sim}{B}(\underset{\sim}{x}))\Big|_{x=0}}$

$$\cancel{\left[\underset{\sim}{x}'\cdot\nabla \underset{\sim}{B}(x)\Big|_{x=0}\right]_i = \sum_{l=1}^{3} x_l' \frac{\partial}{\partial x_l} B_i(x_j)\Big|_{x_j=0}}$$

$i = 1,2,3$

x, y, z

Second Term $\qquad \underset{\sim}{x}' \cdot \nabla B_k$

$$\underset{\sim}{F}_i = \sum_{jk} \epsilon_{ijk} \int d^3x' \, \frac{J_j(\underset{\sim}{x}')}{\cancel{c}} \sum_{\ell} x'_\ell \left(\frac{\partial}{\partial x_\ell} B_k \right)$$

$$\epsilon_{ijk} = \begin{cases} \pm 1 & \text{if } i,j,k \text{ are in order} \\ 0 & \text{otherwise} \end{cases}$$

We have previously derived

$$\cancel{\oint} \int d^3x' \; \underset{\sim}{x} \cdot \underset{\sim}{x}' \, \underset{\sim}{J} = -\frac{1}{2\cancel{c}} \int d^3x' \left[\underset{\sim}{x} \times (\underset{\sim}{x}' \times \underset{\sim}{J}) \right]$$

$$= - \underset{\sim}{x} \times \underset{\sim}{m}$$

j^{th} comp $\qquad$ replace $\underset{\sim}{x}$ by ∇B_k

$$\frac{1}{c} \int d^3x' \; \underset{\sim}{x} \cdot \underset{\sim}{x}' J_j = - \left[\underset{\sim}{x} \times \underset{\sim}{m} \right]_j$$

$$\underset{\sim}{F}_i = + \sum_{jk} \epsilon_{ijk} \left[(\underset{\sim}{m} \times \nabla)_j B_k \right]$$

$$\underset{\sim}{F} = (\underset{\sim}{m} \times \nabla) \times \underset{\sim}{B} = \nabla \, \underset{\sim}{m} \cdot \underset{\sim}{B} - \underset{\sim}{m} \, \cancel{\nabla \cdot \underset{\sim}{B}}^{\,0}$$

$$\underset{\sim}{F} = \nabla (\underset{\sim}{m} \cdot \underset{\sim}{B}) \qquad \text{assumes } \underset{\sim}{m} \text{ is constant } \underset{\sim}{m} \,\&\, \underset{\sim}{B}$$

$$\boxed{\underset{\sim}{B} = -\frac{\mu_0 \underset{\sim}{m}}{4\pi} \cdot \nabla \frac{\underset{\sim}{x}}{|\underset{\sim}{x}|^3} = \frac{\mu_0}{4\pi}\left[\frac{3\underset{\sim}{x}\,\underset{\sim}{x}\cdot m - \underset{\sim}{m}\, x^2}{|\underset{\sim}{x}|^5}\right]}$$

same as $\underset{\sim}{E}$ field due to dipole $\underset{\sim}{p}$

<u>The total torque</u>

$$\underset{\sim}{N} = \int d^3x' \; \underset{\sim}{x}' \times \underset{\sim}{F}$$

$$= \int d^3x' \; \underset{\sim}{x}' \times \frac{1}{\cancel{c}} \left(\underset{\sim}{J}^{(x')} \times \underset{\sim}{B} \right)$$

$$\underset{\sim}{B} = \underset{\sim}{B}(0) + \cancel{\underset{\sim}{B}(} \; \underset{\sim}{x}' \cdot \nabla B$$

Lowest order term

$$\underset{\sim}{N} = \int d^3x' \; \frac{1}{\cancel{c}} \, \underset{\sim}{x}' \times \left(\underset{\sim}{J}(x') \times \underset{\sim}{B}(0) \right)$$

$$= \int d^3x' \; \frac{1}{\cancel{c}} \left[\underset{\sim}{J} \; \underset{\sim}{x}' \cdot \underset{\sim}{B} - \underset{\sim}{B} \; \underset{\sim}{x}' \cdot \underset{\sim}{J} \right]$$

$(\nabla' \cdot \underset{\sim}{J} \doteq 0)$

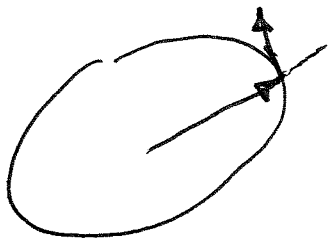
$$\int d^3\underset{\sim}{x}' \; \underset{\sim}{x}' \cdot \underset{\sim}{J} = 0 \qquad \int d^3x \, \tfrac{1}{2} \nabla' x'^2 \cdot \underset{\sim}{J}$$

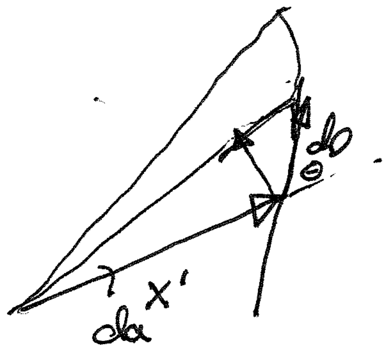
$$= -\int d^3x' \, \tfrac{1}{2} x'^2 \, \nabla' \cdot \underset{\sim}{J} = 0$$

$$\underset{\sim}{N} = \int d^3x' \, \frac{\underset{\sim}{J}}{\cancel{c}} \, \underset{\sim}{x}' \cdot \underset{\sim}{B} = \underset{\sim}{m} \times \underset{\sim}{B}$$

7

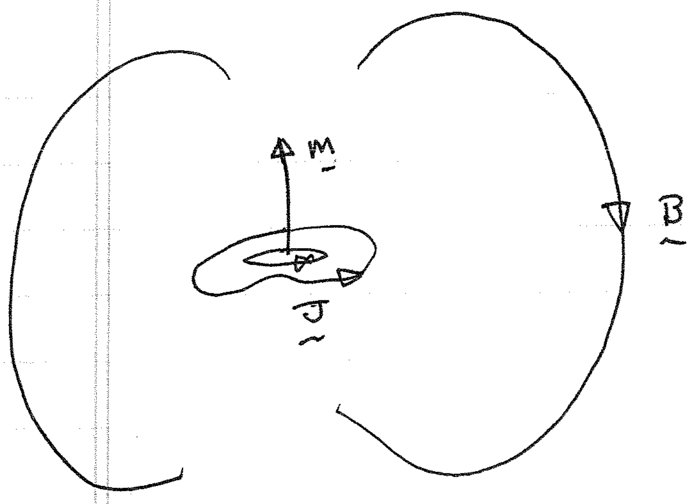
derive $\underset{\sim}{m}$ for a planar loop





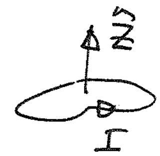
$$da = \frac{x' d\ell_{\perp}}{2}$$

$$|\underset{\sim}{m}| = I \oint da$$



IF $\underset{\sim}{x} \parallel m$ $\quad \cancel{\underset{\sim}{B} \parallel m} \quad \underset{\sim}{B} \cdot \underset{\sim}{m} > 0$

if $\underset{\sim}{x} \perp m$ $\quad \underset{\sim}{B} \cdot \underset{\sim}{m} < 0$

Evaluate $\underset{\sim}{m}$ for a planar current loop 

$$\underset{\sim}{m} = \frac{1}{2} \int \underset{\sim}{x}' \times d\underset{\sim}{\ell}' I = \hat{z} I A$$

Force on a dipole in an external Field

$$\underset{\sim}{B}_{ext}(\underset{\sim}{x}) \simeq \underset{\sim}{B}_{ext}(0) + \underset{\sim}{x} \cdot \nabla \underset{\sim}{B}_{ext} \Big|_{\underset{\sim}{x}=0}$$

$$\underset{\sim}{F} = \nabla \, \underset{\sim}{m} \cdot \underset{\sim}{B}_{ext}(x) \Big|_{\underset{\sim}{x}=0}$$

TORQUE

$$\underset{\sim}{N} = \int d^3\underset{\sim}{x}' \left(\underset{\sim}{x}' \times \left(\underset{\sim}{J}(x') \times \underset{\sim}{B}_{ext}(0) \right) \right) = \underset{\sim}{m} \times \underset{\sim}{B}_{ext}(0)$$

Review of Fields due to localized currents

In general: $\underset{\sim}{B} = \nabla \times \underline{A}$

$$\underline{A} = \frac{\mu_0}{4\pi} \int d^3x' \; \frac{\underset{\sim}{J}(\underline{x}')}{|\underline{x} - \underline{x}'|}$$

For localized current density expand

$$\frac{1}{|\underline{x} - \underline{x}'|} \simeq \frac{1}{|\underline{x}|}\left[1 - \frac{\underline{x}' \cdot \underline{x}}{|\underline{x}|^2} + \cdots\right]$$

$$\underset{\sim}{A} = -\frac{\mu_0}{4\pi} \frac{\underline{x} \times \underline{m}}{|\underline{x}|^3}$$

where $\quad \underset{\sim}{m} = \frac{1}{2} \int d^3x' \; \underline{x}' \times \underset{\sim}{J}(\underline{x}')$

dipole moment

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} = \frac{\mu_0}{4\pi}\left[\frac{3\underline{x}\,\underline{x}\cdot\underline{m} - x^2\underline{m}}{|\underline{x}|^5}\right]$$

Same as E field due to dipole $\underline{p}$

Magnetic Materials

$$\nabla \times \underset{\sim}{B} = \cancel{4\pi}\, \mu_0 \underset{\sim}{J}$$

follow procedure of dielectrics

free current density

induced current densit

$$\underset{\sim}{J} = \underset{\sim}{J}_f + \underset{\sim}{J}_i$$

$$\nabla \cdot \underset{\sim}{E} = 4\pi(\rho_f + \rho_i)$$

note: $\nabla \cdot \underset{\sim}{J}_f = 0$

$\nabla \cdot \underset{\sim}{J}_i = 0$

$$\rho_i = -\nabla \cdot \underset{\sim}{P}$$

polarizatio

$\nabla \cdot \underset{\sim}{J}_i = 0$ implies $\underset{\sim}{J}_i = \nabla \times (\underline{\text{something}})$

<u>answer</u> $\underset{\sim}{J}_i = \nabla \times \underset{\sim}{M}$ ← ~~magnetization density~~

$$\underset{\sim}{M} = \sum_i n_i(x) \langle \underset{\sim}{m}_i \rangle$$

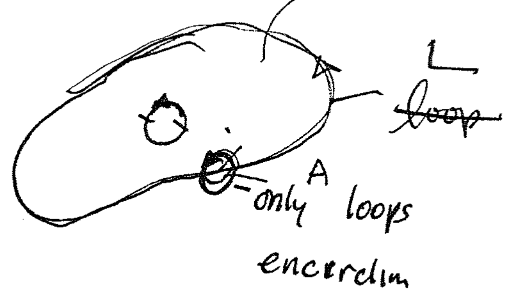
number densit

average magnetic mom

$\underset{\sim}{M}$ = magnetization or magnetic moment density

~~Alternate~~ <u>approach</u>

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \mu_0 I_{\text{enclosed}}$$



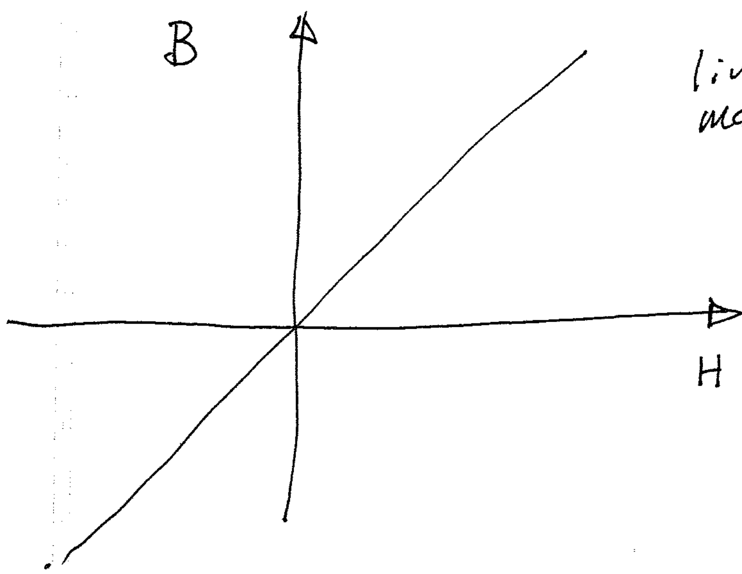
$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} = \mu_0 \left[\underset{\sim}{J}_{free} + \underset{\sim}{J}_{ind} \right]$$

$$= \mu_0 \left[\underset{\sim}{J}_{free} + \nabla \times \underset{\sim}{M} \right]$$

$$\nabla \times \underbrace{\left[\frac{\underset{\sim}{B}}{\mu_0} - \underset{\sim}{M} \right]}_{\text{call this } \underline{H}} = \underset{\sim}{J}_{free}$$

$$\underset{\sim}{B} = \mu_0 (\underset{\sim}{H} + \underset{\sim}{M}) \qquad \text{if} \quad \underset{\sim}{M} \propto \underset{\sim}{H}, \underset{\sim}{B}$$

$$= \mu \underline{H} \qquad \underset{\sim}{H} \sim \mathbb{I} \cdot J_{free}$$



linear material — isotropic etc. etc.

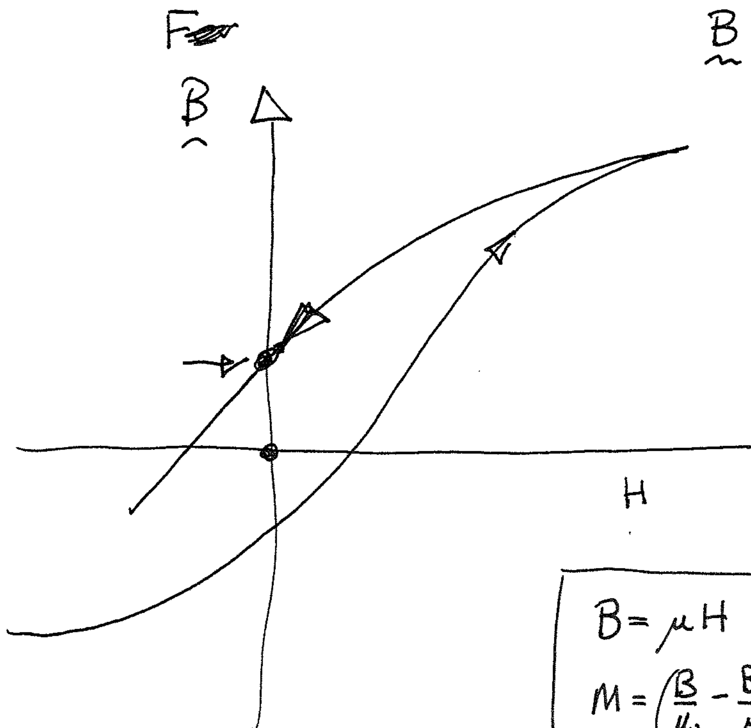
parallel

$0 < \mu < \mu_0$ $\underset{\sim}{B}, \underset{\sim}{H}$ and $\underset{\sim}{M}$ are anti parallel

diamagnetic

$\mu_0 < \mu$ $\underset{\sim}{B}, \underset{\sim}{H}$ & $\underset{\sim}{M}$ are aligned

paramagnetic



$\underset{\sim}{B}$ & $\underset{\sim}{M} \neq 0$

$\underset{\sim}{H} = 0$

ferromagnetic

$$\boxed{\begin{aligned} &B = \mu H \\ &M = \left(\frac{B}{\mu_0} - \frac{B}{\mu}\right) = \frac{(\mu - \mu_0)}{\mu \mu_0} B \end{aligned}}$$

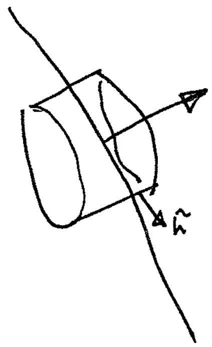
$\mu < \mu_0$

$$\nabla \cdot \underset{\sim}{B} = 0 \qquad \nabla \times \underset{\sim}{H} = \underset{\sim}{J}_{free}$$

Boundary Conditions

#2

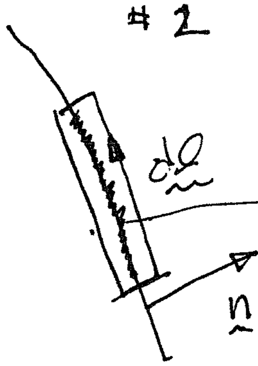
#1



$$\int \underset{\sim}{B} \cdot \hat{n}\, dA \Rightarrow \underset{\sim}{n} \cdot (\underset{\sim}{B}_2 - \underset{\sim}{B}_1) = 0$$

#2

#1



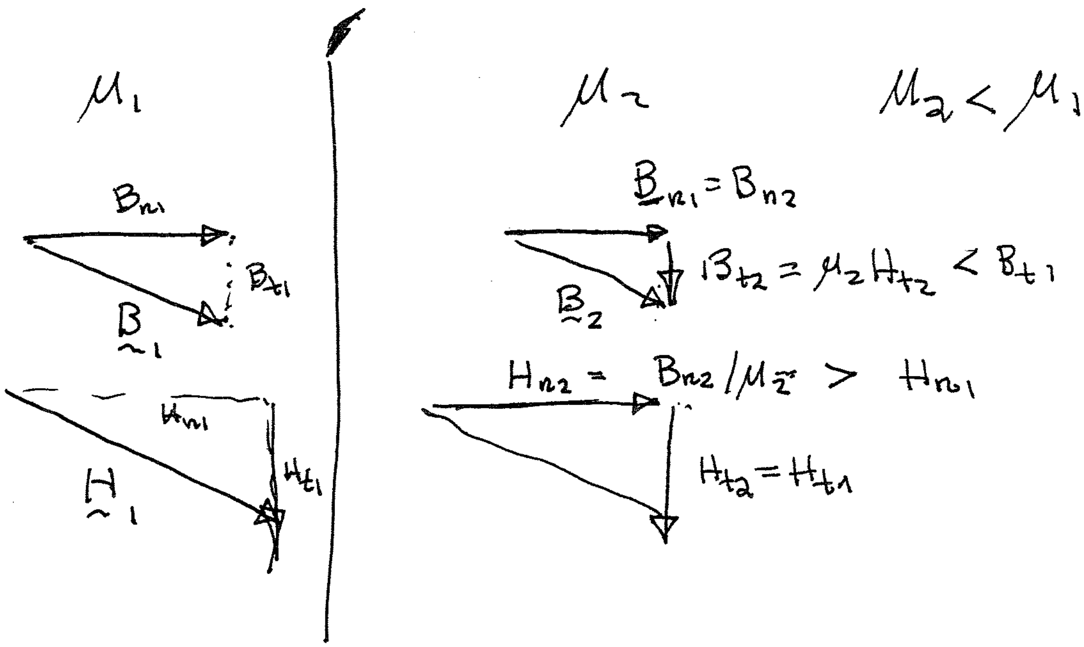
$$\oint \underset{\sim}{H} \cdot d\underset{\sim}{l} = I_{enclosed}$$

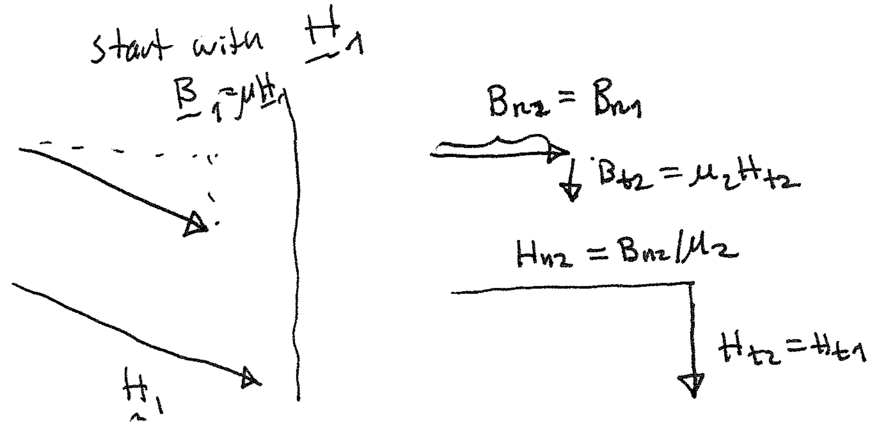
$$(\underset{\sim}{H}_2 - \underset{\sim}{H}_1) \cdot d\underset{\sim}{l}_2 = \underset{\sim}{K}_s \cdot (\underset{\sim}{n} \times d\underset{\sim}{l}_2)$$

$$= (\underset{\sim}{K}_s \times \underset{\sim}{n}) \cdot d\underset{\sim}{l}$$

$$\underset{\sim}{H}_2 - \underset{\sim}{H}_1 = \underset{\sim}{K}_s \times \underset{\sim}{n} \qquad \boxed{\underset{\sim}{K}_s = \underset{\sim}{n} \times (\underset{\sim}{H}_2 - \underset{\sim}{H}_1)}$$

free

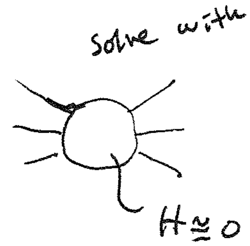




what happens when $\mu_2 >> \mu_1$

if B_{t2} is finite

$H_{t2} \to 0$



$D = \epsilon \underset{\sim}{E}$

$\underset{\sim}{B} = \mu \underset{\sim}{H}$