chapter 20

Magneto Statics

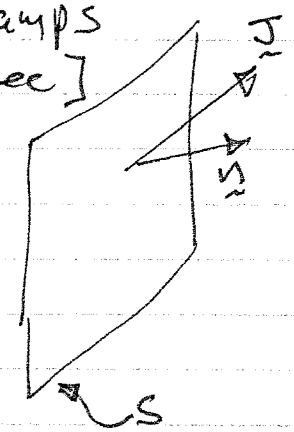
What is current — charge in motion

q move charge

$\mathbf{J}$ = current density (vector) stat amps/cm^2

I = current stat amps [stat coul/sec]

$$I = \int_S \mathbf{J}\cdot\mathbf{n}\, da$$



I = amount of charge passing through surface S per second

for a continuous charge distribution $\rho(\mathbf{x})$ with every charge having same velocity $\mathbf{v}$

$$\mathbf{J} = \rho \mathbf{v}$$

current in conductors ρ_e = electron charge density

ρ_i = positive ion charge density

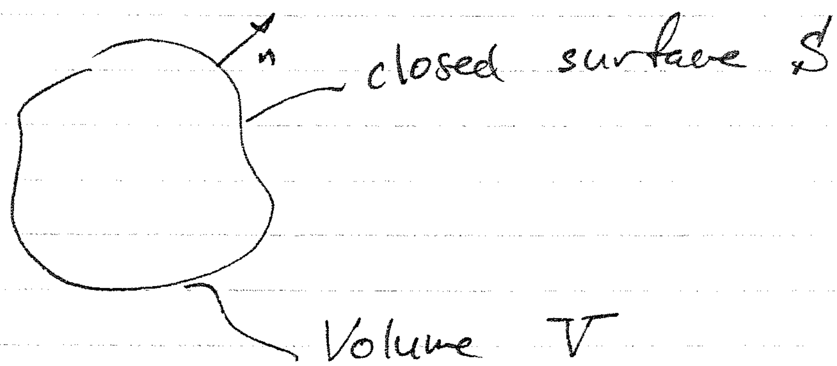
$$\rho_T = \rho_e + \rho_i \simeq 0$$

$$\underset{\sim}{J} = \underbrace{\underset{\sim}{v}_e \rho_e} + \underset{\sim}{v}_i \rho_i \neq 0$$

$\underset{\sim}{v}_i = 0$ metal conduct

$\underset{\sim}{v}_e \neq 0$

Conservation of Charge



$$\frac{dQ}{dt} = \frac{\partial}{\partial t}\int_V d^3x\, \rho \quad * \quad = -\int_S \underset{\sim}{J} \cdot \underset{\sim}{n}\, dA = -I$$

the rate of change of charge inside V is due only to current leaving surface this implies by the divergence theorem

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underset{\sim}{J} = 0$$

Magneto statics ~~$\partial\rho/\partial t = 0$~~

$$\underset{\sim}{J}(\underset{\sim}{x}, t) = \underset{\sim}{J}(\underset{\sim}{x})$$

the current density at any point $\underset{\sim}{x}$ is independent of time

$\partial\rho/\partial t = 0$ $\boxed{\nabla \cdot \underset{\sim}{J} = 0}$

charges are in motion, however there are a large number of charges executing the same motion so that at any instant the ~~charges~~ same amount of charge is passing thru any fixed surface.

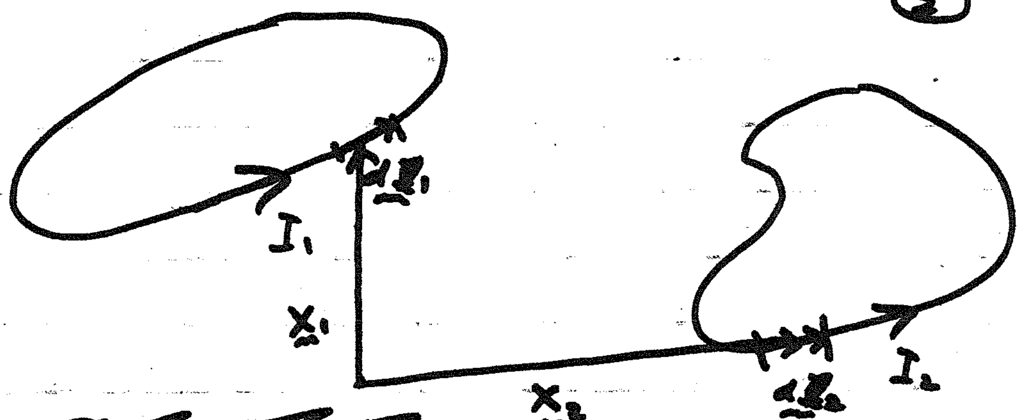
Moving charges (currents exert forces on one another not described by ~~Gauss~~ Col law)

Magnetostatics

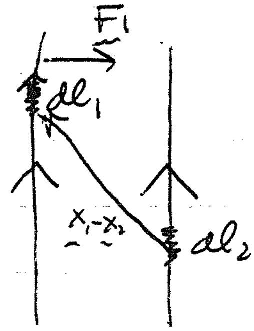
Forces between current carrying conductors

Forces between current carrying conductors : Steady state

$$\frac{dI_1}{dt} = \frac{dI_2}{dt} = 0$$



$$\mathbf{F}_1 = \left(\frac{\mu_0}{4\pi}\right)\frac{I_1 I_2}{c^2}\oint\oint \frac{d\mathbf{l}_1 \times [d\mathbf{l}_2 \times (\mathbf{x}_1 - \mathbf{x}_2)]}{|\mathbf{x}_1 - \mathbf{x}_2|^3}$$



parallel wires attract

F in ~~dynes~~ Newtons

I in statcoulombs/sec = ~~statamperes~~

~~c = 3×10^{10} cm/sec~~ $\mu_0 = 4\pi \times 10^{-7}$

x in ~~cm~~ m

~~It is convenient to define a force field ...~~

~~with $F_1 \propto I_1$~~

must show that $\underset{\sim}{F}_1 = -\underset{\sim}{F}_2$ ~~also~~

$$\underset{\sim}{F}_1 = \frac{I_1 I_2}{c^2} \oint\oint \frac{1}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^3} \Big[d\underset{\sim}{l}_2 \; d\underset{\sim}{l}_1 \cdot (\underset{\sim}{x}_1 - \underset{\sim}{x}_2) - \underbrace{d\underset{\sim}{l}_1 \cdot d\underset{\sim}{l}_2 \, (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)}_{\text{symmetric}} \Big]$$

~~dl_2 ...~~

$$\oint\oint d\underset{\sim}{l}_2 \; d\underset{\sim}{l}_1 \cdot \frac{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)}{|x_1 - x_2|^3} = -\oint\oint d\underset{\sim}{l}_2 \; \underbrace{d\underset{\sim}{l}_1 \cdot \nabla_1 \frac{1}{|x_1 - x_2|}}_{\text{perfect differential}}$$

$$= 0$$

~~$\nabla_1 \frac{1}{|x_1 - x_2|}$~~

$$\underset{\sim}{F}_1 = -\frac{I_1 I_2}{c^2} \oint\oint \frac{(d\underset{\sim}{l}_1 \cdot d\underset{\sim}{l}_2)(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)}{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^3}$$

$$= -\underset{\sim}{F}_2$$

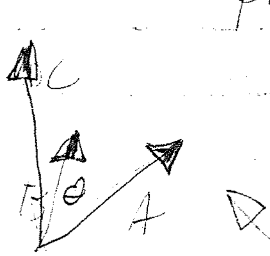
Double ~~cor~~ cross product $[\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C})$

$$\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = \underset{\sim}{B}(\underset{\sim}{A} \cdot \underset{\sim}{C}) - \underset{\sim}{C}(\underset{\sim}{A} \cdot \underset{\sim}{B})$$

$$(\underset{\sim}{A} \times \underset{\sim}{B}) \times \underset{\sim}{C} \quad 6\;7\;8\;9\;10 \quad \overset{?}{=} \underset{\sim}{B}(\underset{\sim}{A} \cdot \underset{\sim}{C}) - \underset{\sim}{A}(\underset{\sim}{B} \cdot \underset{\sim}{C})$$

CROSS product

$$\underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B}$$



$\underset{\sim}{C}$ is perp to both $\underset{\sim}{A}$ & $\underset{\sim}{B}$

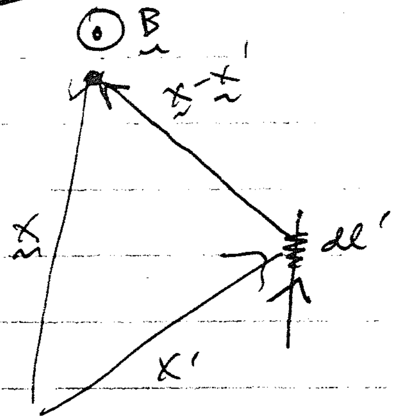
$$|\underset{\sim}{C}| = |\underset{\sim}{A}||\underset{\sim}{B}| \sin\theta$$

$$\underset{\sim}{A} \times \underset{\sim}{B} = -\underset{\sim}{B} \times \underset{\sim}{A}$$

Can define a force field associated with $\mathbf{F}_1$,

$\boxed{\text{do} \mid \text{example}}$

$$\boxed{\mathbf{B}(\mathbf{x}) \equiv \frac{I\mu_0}{4\pi}\oint \frac{d\mathbf{l}'\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}$$



so

$$\boxed{\mathbf{F}_1 = I_1\int d\mathbf{l}_1 \times \mathbf{B}(\mathbf{x}_1)}$$

$\mathbf{B}$ direction of righthand rule

$\mathbf{B}$ in Gauss

define $\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{I\,d\mathbf{l}'\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$

<u>Generalization to continuous current distribution</u>

$$\int I\,d\mathbf{l}' \Longrightarrow \int d^3x'\,\mathbf{J}(\mathbf{x}')$$

$$\boxed{\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d\mathbf{x}'\,\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}$$

$$\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

note again

call $\frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \mathbf{A}$

$$\boxed{\mathbf{B}(\mathbf{x}) = \nabla\times\frac{\mu_0}{4\pi}\int d\mathbf{x}'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}}$$

thus

$\boxed{\nabla\cdot\mathbf{B} = 0}$ First law of magnetostatics $\Longrightarrow$ no magnetic monopoles

$\Longrightarrow$ analogous to $\nabla\times\mathbf{E} = 0$

$$\nabla \times \underline{B} = \nabla \times \{ \nabla \times [A] \}$$

$$= \nabla \, \nabla \cdot [A] - \nabla^2 [A]$$

$$\nabla \cdot [A] = \nabla \cdot \frac{\mu_0}{4\pi} \int d\underline{x}' \, \frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$= \frac{1}{c} \int d\underline{x}' \, \underline{J}(\underline{x}') \cdot \nabla \frac{1}{|\underline{x}-\underline{x}'|}$$

$$= -\frac{1}{c} \int d\underline{x}' \, J(\underline{x}') \cdot \nabla' \frac{1}{|\underline{x}-\underline{x}'|}$$

$$= \frac{1}{c} \int d\underline{x}' \frac{1}{|\underline{x}-\underline{x}'|} \underbrace{\nabla' \cdot J(\underline{x}')}_{0} = 0$$

$$\boxed{\nabla \cdot \underline{A} = 0}$$

but continuity of charge requires

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$$

and since $\frac{\partial \rho}{\partial t} = 0$ in steady state $\nabla \cdot \underline{J} = 0$

$$\Longrightarrow \quad \nabla \cdot [\,] = 0$$

$$\nabla^2 [\,] = \nabla^2 \frac{\mu_0}{4\pi} \int d\underline{x}' \, \frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

Green's function for Poisson Equation

$$= \frac{1}{c} \int d\underline{x}' \, \underline{J}(\underline{x}') \, \underbrace{\nabla^2 \frac{1}{|\underline{x}-\underline{x}'|}}_{-4\pi \, \delta(\underline{x}-\underline{x}')}$$

$$= -\frac{1}{c} \int d\underline{x}' \, \underline{J}(\underline{x}') \, \delta(\underline{x}-\underline{x}') \, 4\pi = -\mu_0 \, \underline{J}(\underline{x})$$

We then have

$$\boxed{\nabla \times \underset{\sim}{B} = \cancel{\tfrac{4\pi}{c}}\, \underset{\sim}{J}(\underset{\sim}{x})}$$

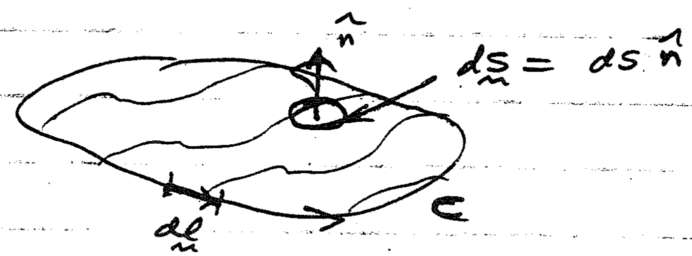
$\mu_0 \underset{\sim}{J}$

Second Law of magnetostatics

~~Amperes~~

$\boxed{\text{Amperes law}}$ — differential form / differential form

Consider a closed surface $\underset{\sim}{S}$ bounded by a curve C



Consider

$$\int d\underset{\sim}{S} \cdot (\nabla \times \underset{\sim}{B}) = \mu_0 \cancel{\tfrac{4\pi}{c}} \int_S d\underset{\sim}{S} \cdot \underset{\sim}{J}(\underset{\sim}{x})$$

$$= \cancel{\tfrac{4\pi}{c}}\, I \qquad \boxed{I = \int_S d\underset{\sim}{S} \cdot \underset{\sim}{J}}$$

where I is the total current passing ~~through~~ S. Using

Stokes theorem

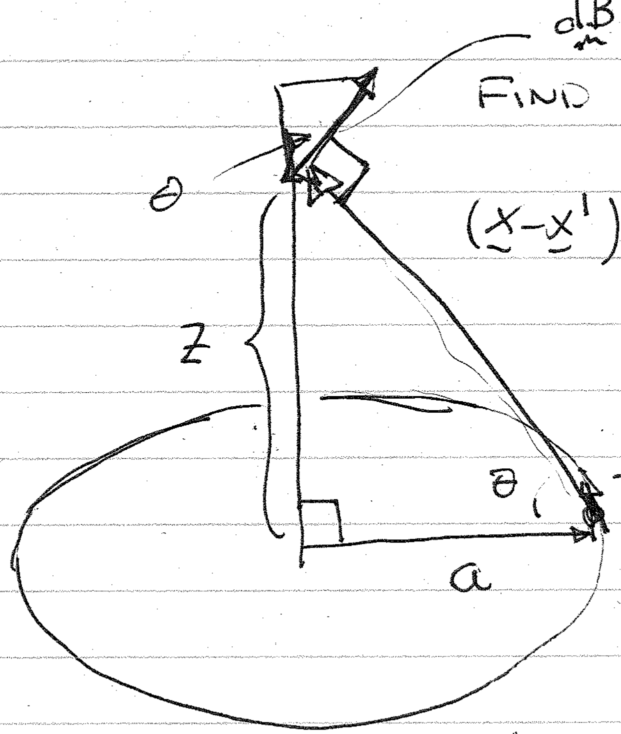
$$\int d\underset{\sim}{S} \cdot \nabla \times \underset{\sim}{B} = \oint_C d\underset{\sim}{l} \cdot \underset{\sim}{B}$$

$$\Rightarrow \boxed{\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \cancel{\tfrac{4\pi}{c}}\, \mu_0 I}$$ Amperes Law.

a^2

$$\underset{\sim}{B} = \frac{\mu_0}{4\pi} \int \frac{I\, d\underset{\sim}{l}' \times (\underset{\sim}{x} - \underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} = \int d\underset{\sim}{B}$$

FIND $\underline{B}$ on axis



$$dB_z = \cancel{\frac{\mu_0}{4\pi}} \cos\theta \, |dB|$$

$$= \cos\theta \, \frac{\mu_0}{4\pi} \frac{I a \, d\theta'}{|x - x'|^2}$$

$$\underset{\sim}{x} = z \hat{z}$$

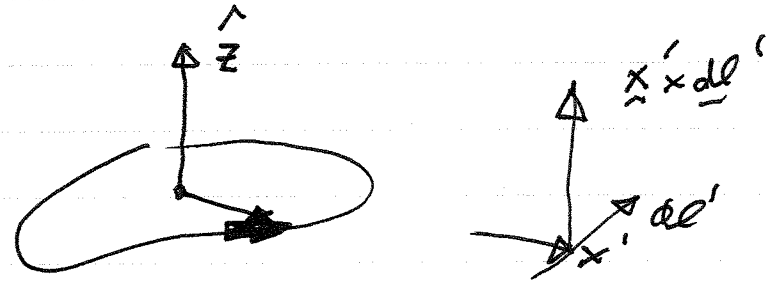
$$\cos\theta = \frac{a}{\sqrt{a^2 + z^2}}$$

$$|x - x'|^2 = (a^2 + z^2)$$

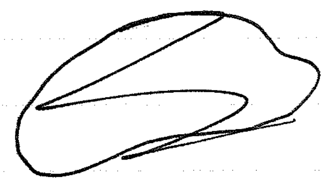
$$dB_z = \frac{a}{\sqrt{a^2+z^2}} \frac{\mu_0}{4\pi} \frac{I}{(a+z^2)} \int d\theta' \qquad B_z = \frac{\mu_0 I}{2} \frac{a^2}{(a^2+z^2)^{3/2}}$$

note $B_z \sim \dfrac{\mu_0 I a^2}{2 z^3}$ as $z \to \infty$

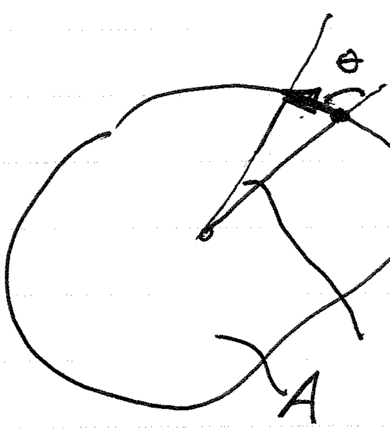
$$\underset{\sim}{m} = \frac{I}{2} \oint \underset{\sim}{x}' \times d\underset{\sim}{\ell}'$$

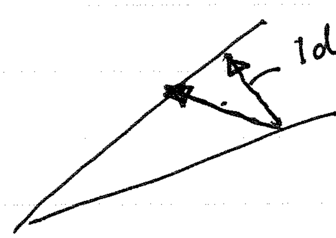


$$\underset{\sim}{m} = \hat{z}\, \frac{I}{2} \int |\underset{\sim}{x}' \times d\underset{\sim}{\ell}'|$$



$$|x' \times d\ell'| = |x'|\,|d\ell'| \sin\theta$$

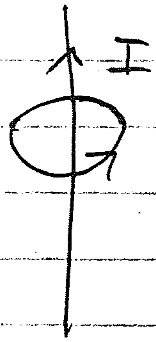




$|d\ell'| \sin\theta = r' d\varphi$

$$\int |\underset{\sim}{x}' \times d\ell'| = \int_0^{2\pi} r'^2(\varphi)\, d\phi = 2A$$

Magnetic Field from Current Carrying Wire



$$\oint \underset{\sim}{B}\cdot d\underset{\sim}{l} = B\, 2\pi r = \frac{4\pi}{c} I$$

Redo

$\underset{\sim}{B}$ direction
right hand rule

$$\boxed{B = \frac{2I}{cr}}$$

$$2\pi r B_\theta = \mu_0 I$$

$$B_\theta = \frac{\mu_0 I}{2\pi r}$$

Vector Potential

Since $\nabla\cdot\underset{\sim}{B} = 0$, can write

$$\underset{\sim}{B} = \nabla\times\underset{\sim}{A}$$

where $\underset{\sim}{A}$ is the vector potential. From previous expression

$$\underset{\sim}{A} = \frac{\mu_0}{4\pi}\int d\underset{\sim}{x}'\,\frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|} + \nabla\psi$$

where ψ is an arbitrary scalar function. ~~[illegible]~~ either.

Since

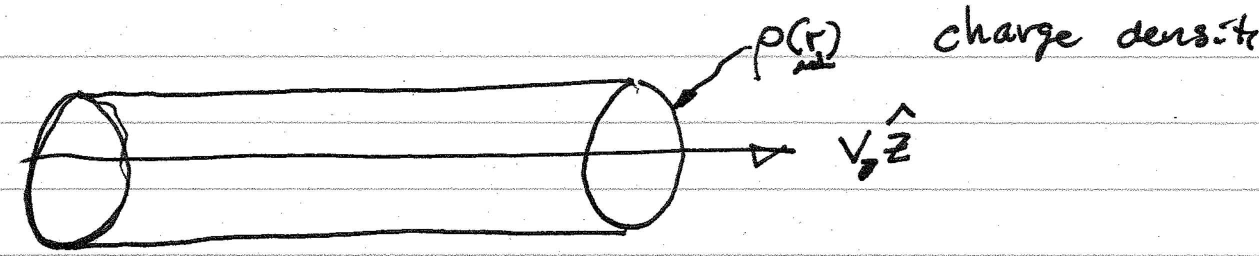
$$\underset{\sim}{B} = \nabla\times\underset{\sim}{A}$$

Since $\nabla\cdot\underset{\sim}{B} = 0$
Really only
2-indepen comp.

is independent of ψ, $\underset{\sim}{A}$ can not be uniquely defined.

Can consider a transformation

Suppose we have a beam



$\nabla^2\phi = -\rho/\epsilon_0 \Rightarrow \phi(\underline{x}_\perp)$

2D problem

$\nabla^2\underline{A} = -\mu_0\rho\underline{V}$

$A_z(\underline{x}_\perp) = V\mu_0\epsilon_0\phi$

What is the force on a ~~the~~ individual charge in the beam?

$\underset{\sim}{F} = +q(\underset{\sim}{E} + \underset{\sim}{V}\times\underset{\sim}{B})$

$\underset{\sim}{E} = -\nabla\phi$

$\underset{\sim}{B} = \nabla\times(\hat{z}A_z) = -\hat{z}\times\nabla_\perp A_z$

$\underset{\sim}{F} = q(-\nabla\phi + V\hat{z}\times(-\hat{z}\times\nabla_\perp A_z)) = -q\nabla_\perp(\phi - V A_z)$

$$A_z = v\mu_0\epsilon_0\phi$$

$$\underset{\sim}{F} = -q\nabla\left(\phi - v^2\epsilon_0\mu_0\phi\right)$$

$$\epsilon_0\mu_0 = \frac{1}{c^2} \qquad \underset{\sim}{F} = -q\frac{1}{\gamma^2}\nabla\phi$$

$$\gamma^2 = \frac{1}{1-v^2/c^2}$$

$$\boxed{\underset{\sim}{A} \Rightarrow \underset{\sim}{A} + \nabla \psi}$$ Gauge transformation

which leave $\underset{\sim}{B}$ unchanged. Can choose $\nabla \cdot \underset{\sim}{A}$ to be ~~what~~ whatever we want.

$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}$$

$$\nabla \times (\nabla \times \underset{\sim}{A}) = \nabla \nabla \cdot \underset{\sim}{A} - \nabla^2 A = \frac{4\pi}{c} \underset{\sim}{J}$$

Coulomb Gauge

We choose $\nabla \cdot \underset{\sim}{A} = 0$ so

$$\boxed{\nabla^2 \underset{\sim}{A} = - \mu_0 \underset{\sim}{J}}$$

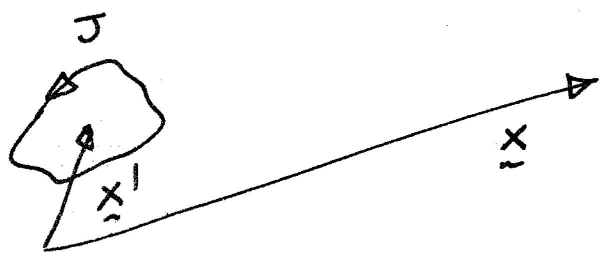
the ~~S~~olution is just

$$\boxed{\underset{\sim}{A} = \frac{\mu_0}{4\pi} \int d\underset{\sim}{x}' \frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}}$$

In this case $\psi = \text{const.}$

Fields due to a localized current distribution (5.6)

$$\underset{\sim}{A} = \cancel{\frac{1}{c}}\,\frac{\mu_0}{4\pi}\int d^3x' \, \frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$



$$\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \simeq \frac{1}{|\underset{\sim}{x}|} + \frac{\underset{\sim}{x}\cdot\underset{\sim}{x}'}{|\underset{\sim}{x}|^3} + \cdots$$

$$\underset{\sim}{A} = \cancel{\frac{1}{c}}\,\frac{\mu_0}{4\pi}\frac{1}{|\underset{\sim}{x}|}\int d^3x' \, \underset{\sim}{J}(\underset{\sim}{x}') + \frac{\mu_0}{4\pi}\frac{\underset{\sim}{x}\cdot}{\cancel{c}|\underset{\sim}{x}|^3}\int d^3x' \, \underset{\sim}{x}' \underset{\sim}{J}(x') + \cdots$$

FIRST TERM VANISHES DUE TO conservation ~~continuity~~ of charge

$$\nabla' \cdot \underset{\sim}{J}(\underset{\sim}{x}') = 0$$

$$\int_V d^3x' \, \underset{\sim}{J}(\underset{\sim}{x}') = -\int_V d^3x' \, \underset{\sim}{x}' \, \nabla' \cdot \underset{\sim}{J}(\underset{\sim}{x}') = 0$$

by parts δ_{ij}

$$\int d^3x' \, J_i = -\int_V d^3x_i' \frac{\partial}{\partial x_j'} J_j = \int d^3x \, J_j \frac{\partial}{\partial x_j'} x_i'$$

~~NOW~~

SECOND TERM will become

$$\mathbf{A} = - \frac{1}{|\mathbf{x}|^3} \frac{\mathbf{x} \times \mathbf{m}\, \mu_0}{4\pi}$$

where $\mathbf{m} = \frac{1}{2\cancel{c}} \int d^3x' \; \mathbf{x}' \times \mathbf{J}(\mathbf{x}')$

magnetic moment

MUST SHOW

$$-\frac{1}{2} \int d^3x' \; \mathbf{x} \times (\mathbf{x}' \times \mathbf{J}) \overset{?}{=} \int d^3x' \; \mathbf{x} \cdot \mathbf{x}' \, \mathbf{J}(\mathbf{x}')$$

USE DOUBLE CROSS

$$-\frac{1}{2} \int d^3x' \left[\mathbf{x}' \, \mathbf{x} \cdot \mathbf{J} - \mathbf{x} \cdot \mathbf{x}' \, \mathbf{J}(\mathbf{x}') \right]$$

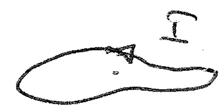
now ~~consider tensor~~ SHOW

$$\int d^3x' \; \mathbf{x}' \, \mathbf{J} \cdot \mathbf{x} = - \int d^3x \; \mathbf{x} \cdot \mathbf{x}' \, \mathbf{J}$$

Magnetic Moment

$$\underset{\sim}{m} = \frac{1}{2} \int d^3x' \; \underset{\sim}{x}' \times \underset{\sim}{J}(x')$$

calculate $\underset{\sim}{m}$ for a loop of current



define magnetic moment density

$$\underset{\sim}{M}(\underset{\sim}{x}') = \frac{1}{2} \underset{\sim}{x}' \times \underset{\sim}{J}(\underset{\sim}{x}')$$

LIKE POLARIZATION $\underset{\sim}{P}$

$$\underset{\sim}{m} = \int d^3x' \; \underset{\sim}{M}(\underset{\sim}{x}')$$

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} = \nabla \times \left(- \frac{\underset{\sim}{x} \times \underset{\sim}{m}}{|\underset{\sim}{x}|^3} \right) \frac{\mu_0}{4\pi}$$

$$= \cancel{- \frac{1}{|x|^3} \nabla \times (\underset{\sim}{x} \times \underset{\sim}{m}) + \nabla \frac{1}{|x|^3} \times (\underset{\sim}{x} \times \underset{\sim}{m})}$$

$$= - \frac{\mu_0}{4\pi} \nabla \times \left[\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|^3} \times m \right] = - \frac{\mu_0}{4\pi} \underset{\sim}{m} \cdot \nabla \frac{\underset{\sim}{x}}{|\underset{\sim}{x}|^3}$$

$$+ \frac{\mu_0}{4\pi} \underset{\sim}{m} \; \underbrace{\nabla \cdot \frac{\underset{\sim}{x}}{|\underset{\sim}{x}|^3}}$$

$\neq 0$

<u>Second term</u>

consider the tensor

$$\int d^3x' \, \underset{\sim}{x}' \, \underset{\sim}{J}(\underset{\sim}{x}') = \int d^3x' \, x_i' \, J_j(x')$$

$$= \int d^3x' \, x_i' \sum_k J_k(\underset{\sim}{x}') \underbrace{\frac{\partial}{\partial x_k'} x_j'}_{\delta_{kj}}$$

do by parts

$$= -\int d^3x' \, x_j' \sum_k \frac{\partial}{\partial x_k'} \left[x_i' J_k(\underset{\sim}{x}') \right]$$

$$= -\int d^3x' \, x_j' \sum_k \left[\underbrace{\frac{\partial x_i'}{\partial x_k'}}_{\delta_{ik}} J_k(\underset{\sim}{x}') + x_i' \underbrace{\frac{\partial}{\partial x_k'} J_k}_{0 \;\; \nabla' \cdot \underset{\sim}{J} = 0} \right]$$

$$= -\int d^3x' \, x_j' \, J_i(\underset{\sim}{x}')$$

$\cancel{\int d^3x' \, x_j' \, J_i(x')}$