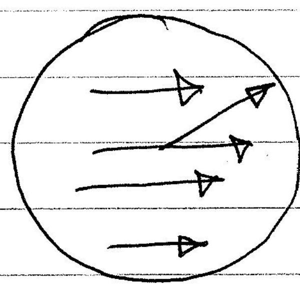
Chapter 18

what is the polarizability

$$P_z = 3\epsilon_0 \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} E_0$$

Alternative Problem:



Polarized sphere of radius a

Polarization density P_z

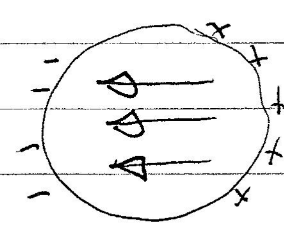
OLD RESult

applied field

effect of P_z

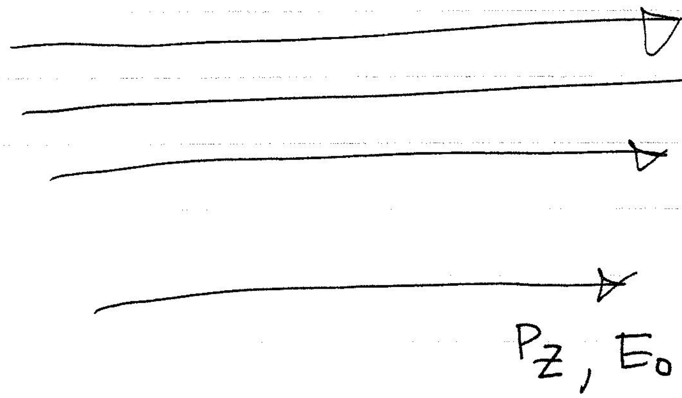
$$E_i = E_0 - \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} E_0$$

$$\boxed{E_i = E_0 - \frac{P_z}{3\epsilon_0}}$$

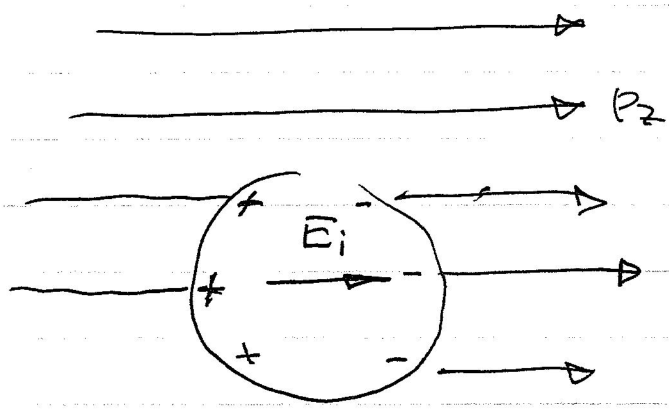


electric field induced by P_z

CONSIDER UNIFORM POLARIZATION Density and field



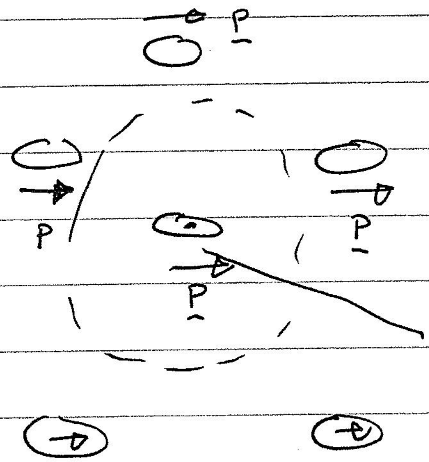
Now Remove a sphere of radius a



$$E_i = E_0 + \frac{P_z}{3\epsilon_0}$$

Field is enhanced

microscopic vs macroscopic field



dipole moments

$P_z = N p_z \qquad p_z = \gamma \epsilon_0 E_{i}$ ~~micro~~

micro field

is applied field

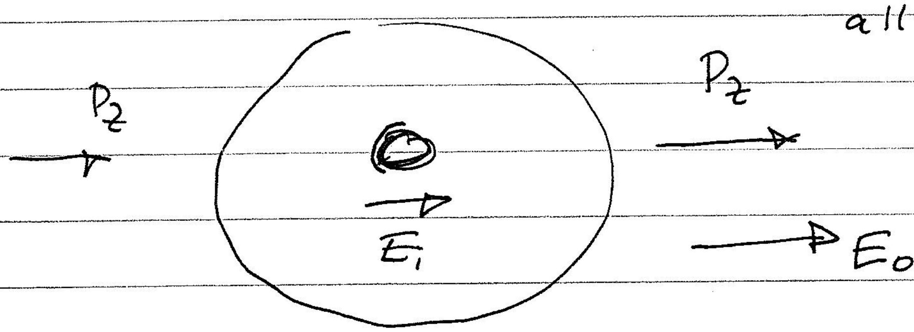
+ field of all other

dipoles

E_i ~~micro~~ is

Replace with

$\longrightarrow P_z$ uniform polarization density representin all othe dipoles



$$E_i = E_0 + \frac{P_z}{3\epsilon_0}$$

~~E_i~~ $P_z = \gamma N \epsilon_0 E_i$

$$\gamma N \epsilon_0 E_i = P_z = \gamma N \epsilon_0 E_0 + \frac{1}{3} \gamma N P_z$$

$$P_Z = \frac{\gamma N \epsilon_0}{1 - \frac{1}{3}\gamma N} E_0$$

$$\chi = \frac{\gamma N}{1 - \frac{1}{3}\gamma N}$$

Clausius – Mossotti Relation

for low density $\chi \propto N$

for high density $\cancel{\chi \propto} N$

Simple model of Polarizability

$$\underset{\sim}{p} = e\underset{\sim}{x}$$

x = displacement

$$\ddot{\underset{\sim}{x}} = -\omega_0^2 \underset{\sim}{x} + \frac{e}{m}\underset{\sim}{E}$$

assume $\quad \underset{\sim}{E} = \mathrm{Re}\left\{ e^{-i\omega t} \hat{\underset{\sim}{E}} \right\}$

$\underset{\sim}{x} = \mathrm{Re}\left\{ e^{-i\omega t} \hat{\underset{\sim}{x}} \right\}$ complex phasor amplitudes

then $\quad \hat{\underset{\sim}{x}} = \frac{e}{m}\hat{\underset{\sim}{E}} \frac{1}{\omega_0^2 - \omega^2}$

$$\hat{\underset{\sim}{p}} = e\hat{\underset{\sim}{x}}$$

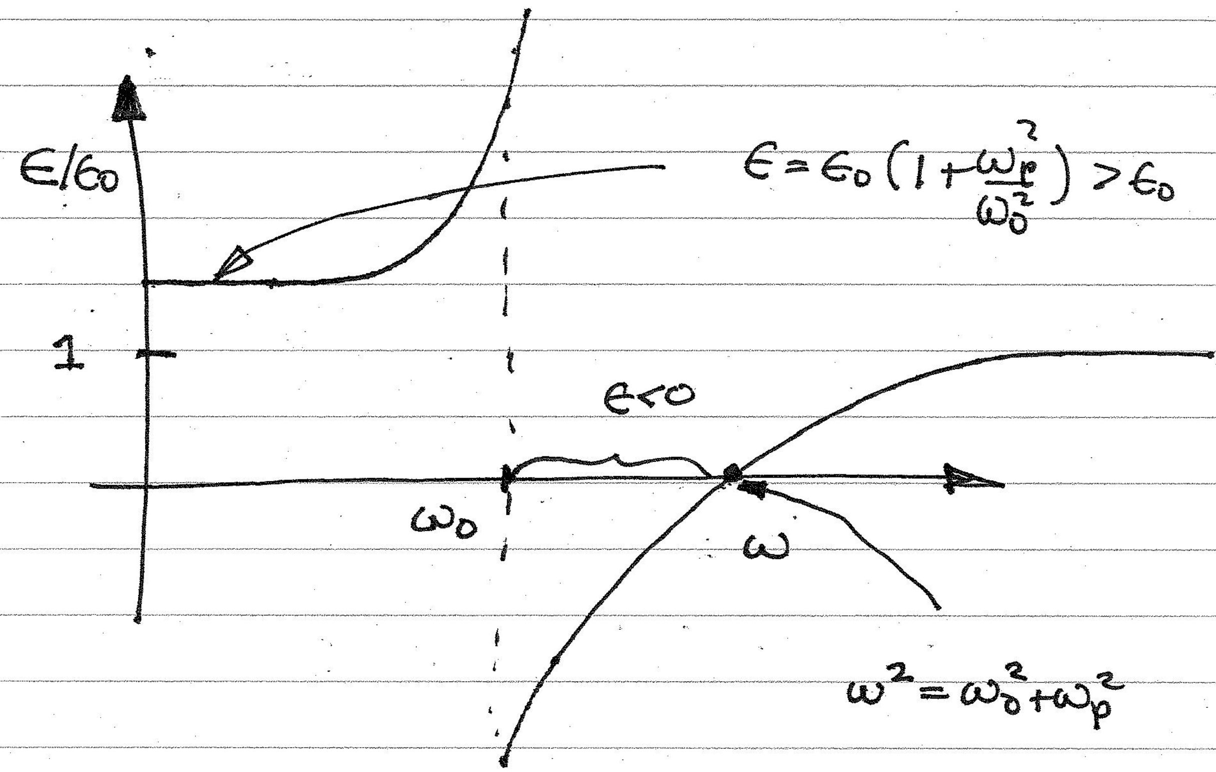
Low density

$$\hat{\underset{\sim}{P}} = N\hat{\underset{\sim}{p}} = \frac{Ne^2}{m} \frac{\hat{\underset{\sim}{E}}}{\omega_0^2 - \omega^2}$$

$$\hat{\underset{\sim}{D}} = \epsilon_0 \hat{\underset{\sim}{E}} + \hat{\underset{\sim}{P}} \qquad \epsilon = \epsilon_0\left(1 + \frac{Ne^2}{m\epsilon_0}\frac{1}{\omega_0^2 - \omega^2}\right)$$

$$= \epsilon \hat{\underset{\sim}{E}}$$

$$\epsilon = \epsilon_0\left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}\right) \qquad \omega_p^2 = \frac{Ne^2}{m\epsilon_0}$$



~~Supp~~

suppose Free electrons $\omega_0^2 \to 0$

$$\epsilon = 1 - \omega_p^2/\omega^2$$

add losses $\ddot{x} + 2\nu\dot{x} = -\omega_0^2 x$

$$-(\omega^2 + i\nu\omega)$$

$$\omega^2 \to \omega^2 + 2i\nu\omega$$

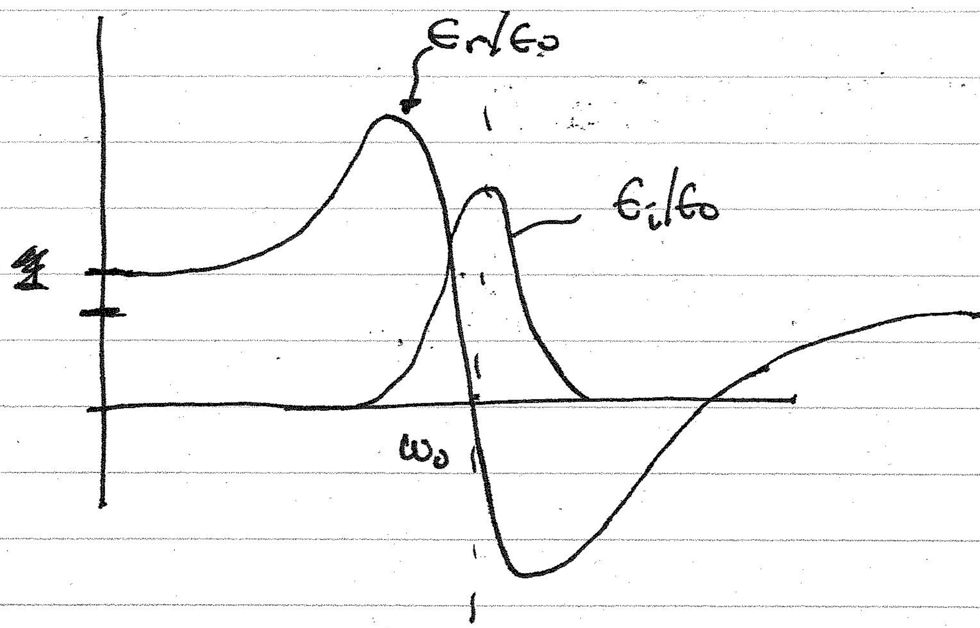
$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - (\omega^2) - 2i\nu\omega}$$

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{\omega_p^2 \left[\omega_0^2 - \omega^2 + 2i\nu\omega\right]}{|\omega_0^2 - \omega^2|^2 + 4\nu^2\omega^2}$$

ϵ_r = even functio

$\epsilon = \epsilon_r + i\,\epsilon_i$

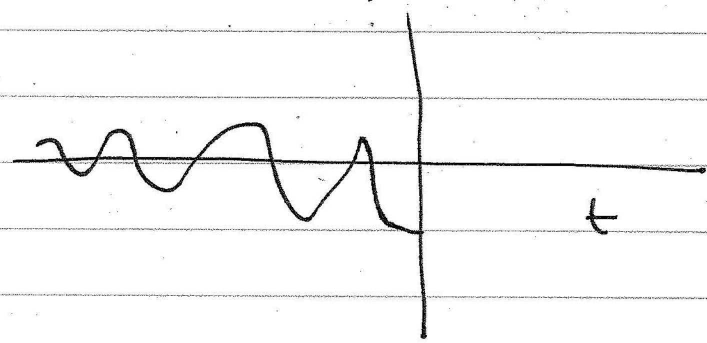
ϵ_i = odd functia > 0 $\omega > 0$



Energy stored in loss less dielectric

$$\nabla \cdot \underset{\sim}{D} = \rho_{free}$$

suppose ρ_{free} ~~oscillates~~ builds up over time



Work done to assemble ρ_{free}

$$W = -\int_{-\infty}^{t} dt' \int d^3x\, \underset{\sim}{J}_{free} \cdot \underset{\sim}{E}$$

Suppose fields are still electrostatic

$$\underset{\sim}{E} = -\nabla \phi$$

$$W = -\int_{-\infty}^{t} dt' \int d^3x\, \phi\, \nabla \cdot \underset{\sim}{J}_{free}$$

$$= -\int_{-\infty}^{t} dt' \int d^3x\, \phi \left(-\frac{\partial \rho_{free}}{\partial t}\right) = \int_{-\infty}^{t} \cancel{dt' \int d^3x\, \phi \left(\frac{\partial}{\partial t} \nabla \cdot \frac{\partial D}{\partial t}\right)}$$

$$W = -\int_{-\infty}^{t} dt' \int d^3x \; \mathbf{E} \cdot \mathbf{J}_{free}$$

$$\mathbf{J}_{free} = \nabla \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D}$$

$$W = +\int_{-\infty}^{t} dt' \int \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \mp \mathbf{E} \cdot \nabla \times \mathbf{H} \right) d^3x$$

note: $\mathbf{E} \cdot \nabla \times \mathbf{H} = \nabla \cdot \mathbf{H} \times \mathbf{E} + \mathbf{H} \cdot \nabla \times \mathbf{E}$

$$W = \int_{-\infty}^{t} dt' \int \left[\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{H} \times \mathbf{E}) \right] d^3x$$

take $S \to \infty$

$$W = \int_{-\infty}^{t} dt' \left[\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right]$$

$$W = -\int_{-\infty}^{t} dt' \, \phi \, \nabla \cdot \left(-\frac{\partial P}{\partial t}\right)$$

$$= \int_{-\infty}^{t} dt' \int \phi \, \nabla \cdot \frac{\partial P}{\partial t} \, d^3x$$

do by parts again

$$= \int_{-\infty}^{t} dt' \int d^3x \, \underset{\sim}{E} \cdot \frac{\partial \underset{\sim}{P}}{\partial t}$$

$$\underset{\sim}{E} = Re\left\{ e^{-i(\omega_r + i\gamma)t} \, \hat{\underset{\sim}{E}} \right\}$$

$$\underset{\sim}{P} = Re\left\{ e^{-i(\omega_r + i\gamma)t} \, \epsilon(\omega_r + i\gamma) \, \hat{\underset{\sim}{E}} \right\}$$

$$W = \int_{-\infty}^{t} dt' \int \underset{\sim}{E} \cdot \frac{\partial \underset{\sim}{P}}{\partial t} \, d^3x$$

$$W = \frac{1}{4} \int_{-\infty}^{t} dt' \int d^3x \Big[\hat{\underset{\sim}{E}}^{*} \cdot -i(\omega_r + i\gamma) \, \epsilon(\omega_r + i\gamma) \, \hat{E} \; e^{2\gamma t'}$$

$$+ \; \hat{\underset{\sim}{E}} \cdot i(\omega_r - i\gamma) \, \epsilon(\omega_r - i\gamma) \, \hat{E}^{*}$$

$$+ \text{ terms oscillate } e^{\pm 2 i \omega_r t + 2\gamma t}$$

$$\langle W \rangle_{\omega_r} = \int d^3x \; \frac{1}{4} \int_{-\infty}^{t} dt' \, |\hat{E}|^2 e^{2\gamma t'}$$

$$\Big[-i(\omega_r + i\gamma)\epsilon(\omega_r + i\gamma) + i(\omega_r + i\gamma)\epsilon(\omega_r + i\gamma) \Big]$$

$$-i\omega_r \epsilon(\omega_r) + \gamma\epsilon(\omega_r) + \omega_r \gamma \frac{\partial \epsilon}{\partial \omega_r} + O(\gamma^2)$$

cancel

$$+i\omega_r \epsilon + \gamma\epsilon(\omega_r) + \omega_r \gamma \frac{\partial \epsilon}{\partial \omega_r} + O(\gamma^2)$$

$$\langle W \rangle_{\omega_r} = \int d^3x \; \frac{1}{4} \frac{|\hat{E}|^2 e^{2\gamma t}}{2\gamma} \, 2\gamma \Big[\epsilon + \omega_r \frac{\partial \epsilon}{\partial \omega_r} \Big]$$

TAKE Limit $\gamma \to 0$

$$\langle W \rangle_{\omega_r} = \frac{1}{4} \int d^3x \; |\hat{E}|^2 \frac{\partial}{\partial \omega_r} \big(\omega_r \epsilon(\omega_r) \big)$$

$$= \frac{1}{2} \int d^3x \; |\hat{E}|^2_{RMS} \frac{\partial}{\partial \omega} \big(\omega_r \epsilon(\omega_r) \big)$$

next I bring in an increment of free charge from infinity

$\delta\rho_f$. To do this work must be done.

no $\frac{1}{2}$ increment

$$\delta W_f = \int d^3x \, \delta\rho_f \, \phi(\underset{\sim}{x})$$

this energy will be stored in E field and medium.

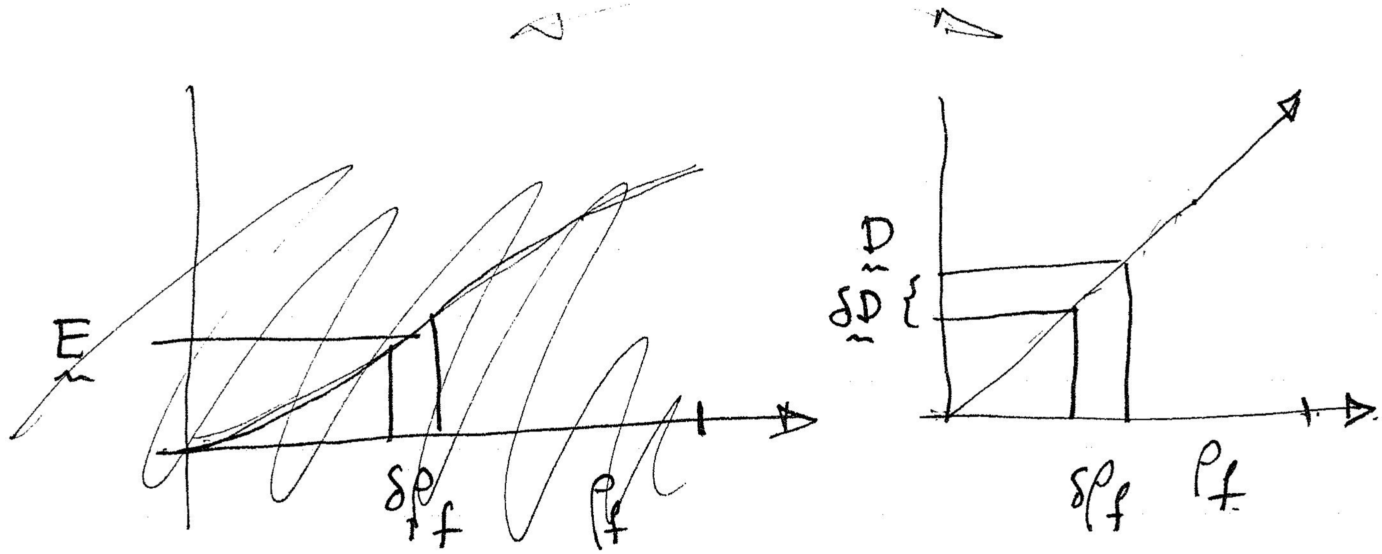
note that due to introduction of $\delta\rho_f$ there will be a change in the displacement δD

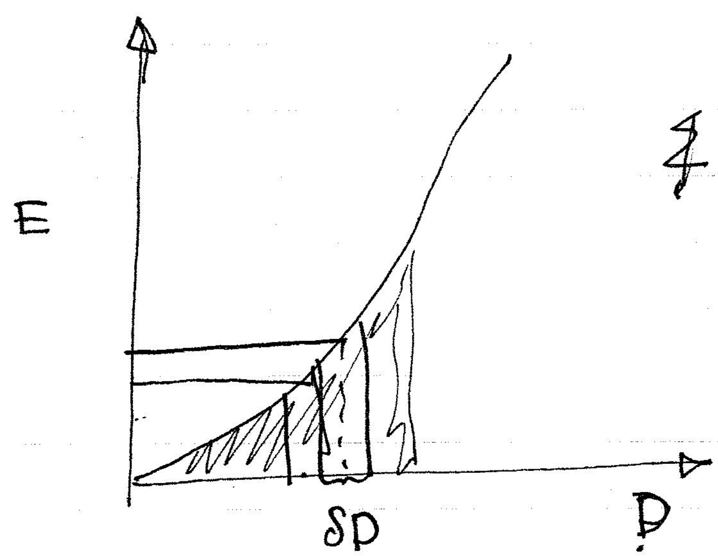
$$\nabla \cdot \delta \underset{\sim}{D} = \cancel{4\pi} \, \delta\rho_f$$

$$\delta W_f = \int d^3x \, \cancel{\tfrac{1}{4\pi}} (\nabla \cdot \delta \underset{\sim}{D}) \, \phi(\underset{\sim}{x})$$

$$= -\int d^3x \, \cancel{\tfrac{1}{4\pi}} \, \delta \underset{\sim}{D} \cdot \nabla \Phi = \int d^3x \, \frac{\delta \underset{\sim}{D} \cdot \underset{\sim}{E}}{\cancel{4\pi}}$$

To assemble final state one & sums up contributions from all increments





$$W_t = \int d^3x \int_0^{D} dD' \cdot E'(D')$$

$$W = \int d^3x \int$$

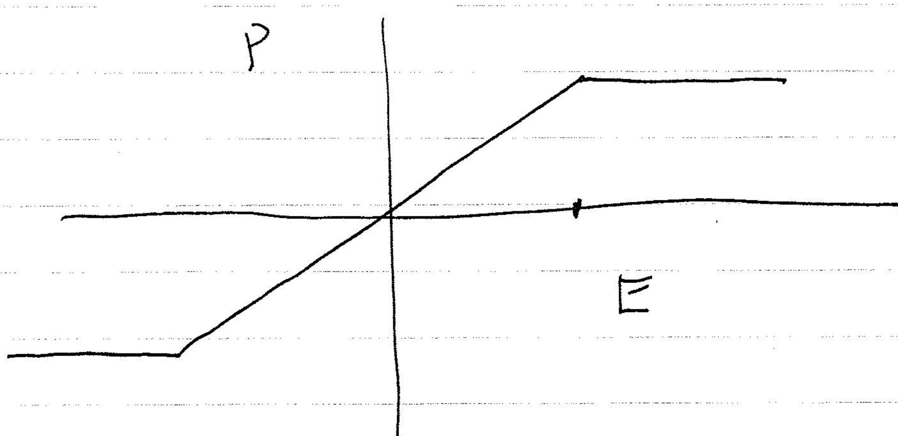
supose Medium is linear

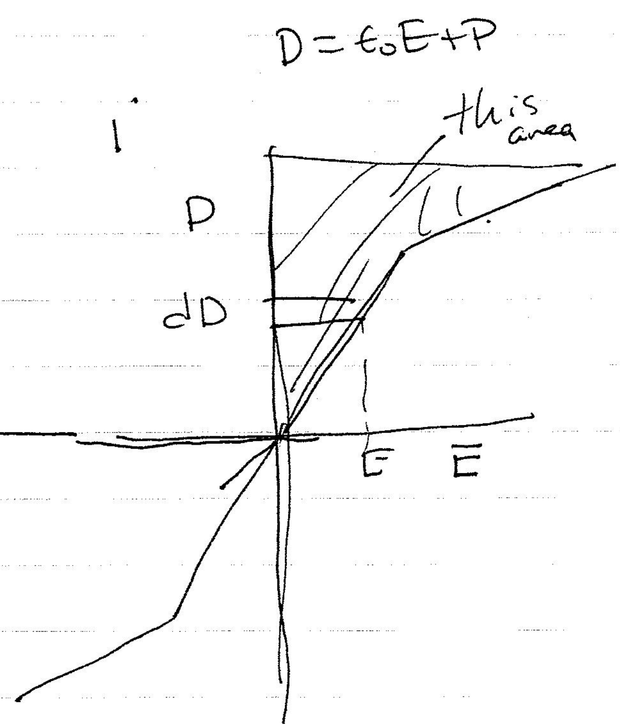
v13 $\quad \epsilon \mathbf{E} = \mathbf{D} \qquad d\mathbf{D} = \epsilon\, d\mathbf{E}$

$$W = \int d^3x \,\frac{\epsilon |\mathbf{E}|^2}{2} \quad \text{or} \quad = \int d^3x\, \frac{\mathbf{D}\cdot\mathbf{E}}{2}$$

Valid for fixed dielectrics

Nonlinear





$$\int_0^{\mathbf{D}} \mathbf{E}\cdot d\mathbf{D} = \mathbf{E}\cdot\mathbf{D} - \int_0^{\mathbf{E}} \mathbf{D}\cdot d\mathbf{E}$$

Electrostatic Energy in Dielectric Media

Energy stored in creating a ~~system~~ charge distribution

$$W = \frac{1}{2}\int \rho(x)\phi(x)\, d^3x$$

what should we put for $\rho(\underset{\sim}{x})$

option # 1 $\quad \rho(x) = \rho_f(x) + \rho_i(x)$

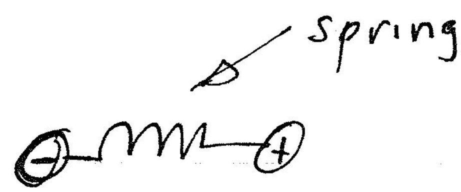
$$\underbrace{\rho(x)}_{\rho}$$

$$\rho_i(x) = -\frac{1}{\cancel{}}\nabla\cdot(\epsilon-\epsilon_0)\underset{\sim}{E}$$

$$\rho(x) = \nabla\cdot\epsilon\underset{\sim}{E} \qquad \rho_f = \nabla\cdot\epsilon\underset{\sim}{E}$$

only counts macroscopic electrostatic energy

Does not count energy stored micro-scopically in medium



dumbell's with springs

energy is stored in spring not included as electro static energy.

alternate procedure consider a system ~~of dielectric~~ with a dielectric such that

$$\underset{\sim}{D} = \epsilon_0 \underset{\sim}{E} + \underset{\sim}{P}$$

$$\nabla \cdot \underset{\sim}{D} = \rho_f$$ ← free charge

$$\underset{\sim}{E} = -\nabla \Phi$$