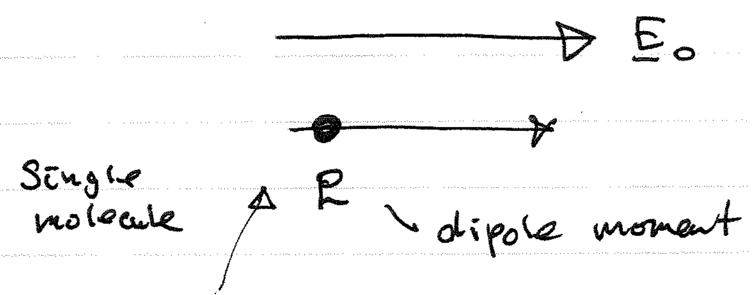
chapter 17

Clausius Mositti Relation

What is the relation between individual molecular polarizabilities and the average electric field

external field



$$\underline{p} = \epsilon_0 \gamma \underline{E}_0$$

γ = molecular polarizability

Low density

Polarization density

$$\underline{P} = N\underline{p}$$

$$\underline{P} = \epsilon_0 N \gamma \underline{E}_0 \qquad \chi = N\gamma$$

Low density

what is χ ?

$$p_m = \epsilon_0 \gamma \, \underset{\sim}{E}_{ext}$$

molecular polarizabili

ext field acting on dipole

if we say $\underset{\sim}{E}_{ext} = \underset{\sim}{E}$ ← average field

$$\underset{\sim}{P} = N p_m = \epsilon_0 \underbrace{N\gamma}_{\chi \,=\, N\gamma} \underset{\sim}{E}$$

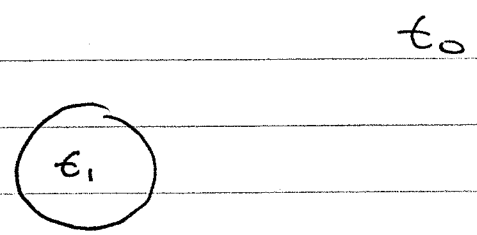
linearly proportional to density

~~(Not~~

accurate for N low

Interesting Cases

1 Outside is vacuum $\epsilon_2 = \epsilon_0$



$$p = 4\pi\epsilon_0 a^3 \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} E_0$$

note if $\epsilon_1 = \epsilon_0$ $p = 0$ as it should

What happens when $\epsilon_1 \gg \epsilon_0$ $\quad p = 4\pi\epsilon_0 a^3 E_0$

independent of ϵ_1 ~~Also true~~

What is $E_i = E_0 - \dfrac{p}{4\pi\epsilon_0 a^3} = \left(1 - \dfrac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0}\right) E_0$

$$E_i = \frac{3\epsilon_0}{\epsilon_1 + 2\epsilon_0} E_0$$

if $\epsilon_1 > \epsilon_0$ $\quad E_i < E_0$ $\quad E_i \to 0$ as $|\epsilon_1| \to \infty$

inside is shielded

Polarization Density inside

$$\cancel{P} \qquad D = \epsilon \underline{E} = \epsilon_0 \underline{E} + \underline{P}$$

$$\underline{P} = (\epsilon - \epsilon_0)\underline{E}$$

Inside $\quad P = (\epsilon_1 - \epsilon_0)E_i = 3\epsilon_0 \dfrac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} E_0$

$$\boxed{\underbrace{\tfrac{4}{3}\pi a^3}_{\text{volume}} P = p \quad \text{dipole moment}}$$

Some materials have $\epsilon < 0$ (plasmas)

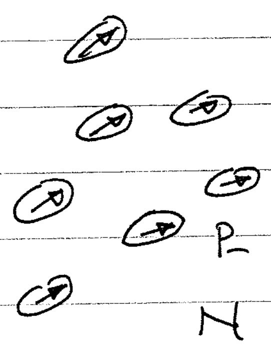
if $\epsilon_1 = -2\epsilon_0$ $\qquad$ P is finite
$E_0 \to 0$

"Resonance"

$$\epsilon_1 = \epsilon_0 \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right) = -2\epsilon_0$$

$$\boxed{\omega^2 = \frac{1}{3}\omega_{pe}^2 \qquad \text{Reduced plasma freq}}$$

Summary



Medium consists of many small dipoles

$p(x)$ dipole moment of dipole at x

$N(\underset{\sim}{x})$ # density of dipoles

Replace by average Polarization density

$\underset{\sim}{P} = N \underset{\sim}{p}$ $\quad \underset{\sim}{P}(x)$ is a smooth function of x

$\rho_i = -\nabla \cdot \underset{\sim}{P}$ average induced charge density

$\rho = \rho_{free} + \rho_{ind}$

$$\nabla \cdot \underbrace{(\epsilon_0 \underline{E} + \underline{P})}_{\underset{\sim}{D}} = \rho_{free}$$

approximation: $\underset{\sim}{P} = \epsilon_0 \chi \underline{E}$ average field

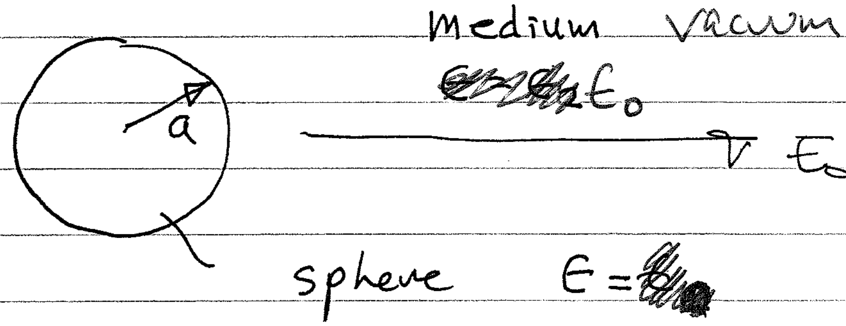
susceptibility

$\underline{D} = \epsilon \underline{E}$

$\epsilon = \epsilon_0 (1 + \chi)$

p

Dielectric Sphere in a uniform medium



$\underset{\sim}{E} = -\nabla\phi \qquad \nabla\cdot\epsilon\underset{\sim}{E} = 0$

$\nabla^2\phi = 0$ inside sphere

$\nabla^2\phi = 0$ outside sphere

$\phi \to -rE_0\cos\theta$

$|r| \to \infty$

ϕ continuous at $r=a$

$\epsilon_0 \frac{\partial\phi}{\partial r}\Big|_{a+0} - \epsilon \frac{\partial\phi}{\partial r}\Big|_{a-0} = 0$ no free surface charge

Outside Sphere $\nabla^2\phi = 0$

$$\phi = \sum_\ell P_\ell(\cos\theta)\left[a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}} \right]$$

as $r \to \infty$ we usualy reject r^ℓ solution $a_\ell = 0$

In this case boundary condition is not $\Phi \to 0$ as $r \to \infty$ rather $\phi \to -E_0 r\cos\theta$

$P_1 = \cos\theta$

$$a_\ell = 0 \qquad \ell \neq 1 \qquad a_1 = -E_0$$

Inside sphere

$$\epsilon \nabla^2 \Phi = 0$$

$$\phi = \sum_\ell P_\ell(\cos\theta)\, c_\ell\, r^\ell$$

at $r = a$ ϕ is continuous (or $E_t \propto \frac{\partial \phi}{\partial \theta}$ is continuous)

$D_r = \epsilon E_r$ is continuous

$$-\left[\overset{\epsilon_0}{(1)} \frac{\partial \phi}{\partial r}\Big|_{r=a+0} - \epsilon \frac{\partial \phi}{\partial r}\Big|_{r=a-0} \right] = 0$$

for each ℓ

o.k.

$$a_\ell a^\ell + b_\ell / a^{\ell+1} = c_\ell a^\ell$$

for each ℓ

$$\ell a_\ell a^{\ell-1} - (\ell+1) b_\ell / a^{\ell+2} = \epsilon \ell c_\ell a^{\ell-1}$$

$$\ell a_\ell a^\ell - (\ell+1) b_\ell / a^{\ell+1} = \epsilon \ell c_\ell a^\ell$$

$$a_1 = -E_0$$

$$b_1 = \frac{E_0(\epsilon-\epsilon_0)}{\epsilon+2\epsilon_0}\, a^3$$

$$C_1 = -E_0 + E_0\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}$$

$$= -\frac{3\epsilon_0 E_0}{\epsilon+2\epsilon_0}$$

$$\phi = \cos\theta\left[\frac{a^3}{r^2}\frac{E_0(\epsilon-\epsilon_0)}{\epsilon+2\epsilon_0} - E_0 r\right] \qquad r>a$$

$$= \cos\theta\left[-\frac{3\epsilon_0 E_0}{\epsilon+2\epsilon_0} r\right] \qquad r<a$$

if $= \epsilon=\epsilon_0$ $\quad \phi = -E_0 r\cos\theta$ extern field calculate $\underset{\sim}{P} = \underset{\sim}{D} - \epsilon_0 \underline{E}$

$P = (\epsilon - \epsilon_0)\underline{E}$

outside $\underset{\sim}{P} = 0$

what is

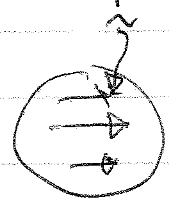
$\underset{\sim}{p} = \int d^3x \, \underset{\sim}{P}$

inside $\underset{\sim}{P} = \epsilon_0 \dfrac{3(\epsilon - \epsilon_0)}{\epsilon + 2\epsilon_0} \underline{E}_0$

~~$p = \int d^3x \, p \ldots$~~

$\underset{\sim}{p} = \frac{4}{3}\pi a^3 \underset{\sim}{P}$

$\underset{\sim}{P} = \text{const}$



$\underset{\sim}{P} = 0$

$\phi_{out} = -\cos\theta \left[rE_0 - \dfrac{p}{4\pi\epsilon_0 r^2} \right]$

Question: Can $\underline{E}_0 = 0$

and $\underset{\sim}{P} \neq 0$

$\omega_p^2 = \dfrac{ne^2}{m\epsilon_0}$

Ans: yes ie $\epsilon = -2\epsilon_0$

dielectric resonance

plasma $\epsilon = \epsilon_0 \left(1 - \dfrac{\omega_p^2}{\omega^2}\right)$

$\epsilon_\wedge = -(\ldots)\epsilon_0 ll$

Show $\quad (\rho_i = -\nabla \cdot \underset{\sim}{P})$

$$\int d^3x \; \underset{\sim}{x} \rho_i = \int d^3x \; -x \nabla \cdot \underset{\sim}{P}$$

$$= \int d^3x \; \underset{\sim}{P} \cdot \nabla \underset{\sim}{x}$$

$$\left(\int d^3x \, \underset{\sim}{P} \cdot \nabla x_j\right) = \sum_j \int d^3x \; P_i \underbrace{\frac{\partial}{\partial x_i} x_j}_{\delta_{ij}} = \int d^3x \, P_j$$