Chapter 16

Lect. 13 Chapter 4 Only 4.3, 4.4, 4.7

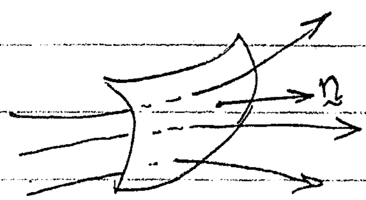
Problems in Electrically Conducting Media

(This is not in the book, but you are responsible for what is presented in lecture on this topic.)

Electrical conductivity:

~~Stat coul/sec~~

$\underset{\sim}{J}$ = current density = $\dfrac{\text{current}}{\text{area}}$ ~~I = current = Abamps~~



$\underset{\sim}{J}$ $\quad I = \int_S \underset{\sim}{J}(\underset{\sim}{r}) \cdot \underset{\sim}{n}\, da$

$= \left(\begin{array}{l}\# \text{ of } \cancel{\text{stat}} \text{ coul.} \\ \text{flowing thru surface } S \text{ per sec.}\end{array}\right)$

Ohm's Law:

In many materials (called linear conductors)

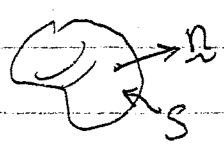
$\underset{\sim}{J} \propto \underset{\sim}{E}$

Can write this as

$\underset{\sim}{J} = \sigma \underset{\sim}{E}$ where σ is the electrical conductivity of the medium.

(We also use σ for surface charge density; so to know what it is consider context.)

In cases where charge distribution does not change with time, for any closed surface S bounding a vol. V



$\oint \underset{\sim}{J} \cdot \underset{\sim}{n}\, da = \left(\begin{array}{l}\text{rate at which charge} \\ \text{leaves } V\end{array}\right) = 0$

This implies

$$\nabla \cdot \underset{\sim}{J} = 0$$

or

$$\nabla \cdot \sigma \underset{\sim}{E} = 0$$

or if σ is not a function of $\underset{\sim}{x}$ and we write $\underline{E} = -\nabla\phi$

$$\nabla^2 \Phi = 0$$

$$\phi = \sum_\ell P_\ell(\cos\theta)\left[a_\ell r^{-\ell} + b_\ell r^{\ell+1}\right]$$

as $r \to \infty$ $\phi = -r\cos\theta E_0$

Thus $b_\ell = 0$ ~~note~~ except $b_1 = -E_0$

$P_1(\cos\theta) = \cos\theta$

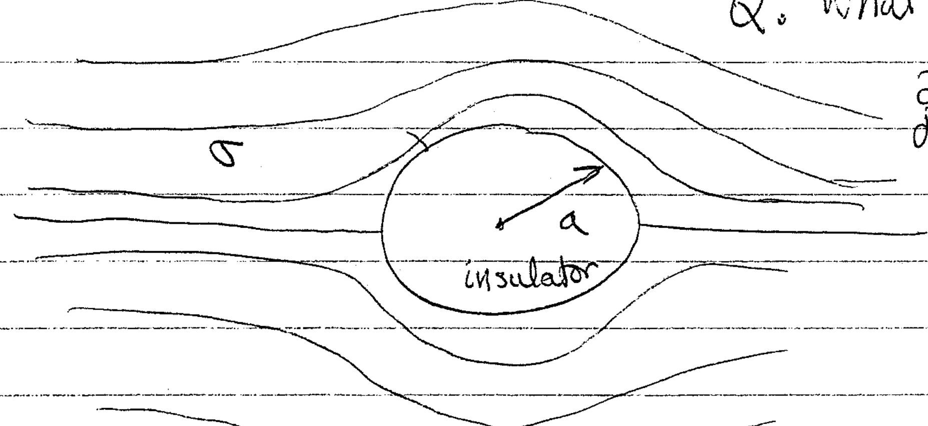
Lect. 14 '98 Ch. 4 only 4.3, 4.4, 4.7

Example — Lect. 7 '03

$\underset{\sim}{E} \cong E_0 \underset{\sim}{z}_0$ for $\rho \gg a$ in a conducting medium of conductivity σ.
Find $\underset{\sim}{J}$ every where if there is a cylindrical hole of radius a cut into the conducting medium

$$\frac{\partial\phi'}{\partial r} = \sum_\ell P_\ell(\cos\theta)\left[-\ell a_\ell r^{-(\ell+1)} + (\ell+1) b_\ell r^\ell\right]$$

Q. What is b.c. at $\rho = a$?



$$\left.\frac{\partial\phi}{\partial r}\right|_a = 0$$

$+2b_1 r^\ell = 0$

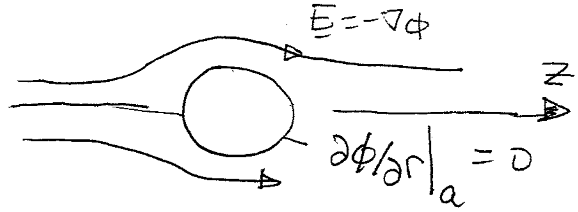
$P_\ell(\cos\theta)$ are orthogonal

$\therefore a_\ell = 0$ except a_1

Boundary condition on $\rho = a$: $\underset{\sim}{j} \cdot \underset{\sim}{n} = 0$ (explain why)

$$\underset{\sim}{J} \cdot \underset{\sim}{n} = \sigma \underset{\sim}{E} \cdot \underset{\sim}{n} = -\sigma \nabla\Phi \cdot \underset{\sim}{n} = -\sigma \left.\frac{\partial\phi}{\partial\rho}\right|_{\rho=a} = 0$$

Problem: $\nabla^2 \Phi = 0$ (with azimuthal symmetry)



Far away

$-\nabla\Phi = \hat{z} E_0$

$\Phi = -zE_0$

$= -r\cos\theta E_0$

$$\Phi = \sum_l (A_l r^l + B_l r^{-(l+1)}) P_l(\cos\theta))$$

$e^{im\varphi}$ $m=0$

as $r \to \infty$ $\Phi = \sum_l A_l r^l P_l(\cos\theta)$

$= -E_0 r\cos\theta$

THIS implies $A_l = 0 \quad l \neq 1$

$A_1 = \cancel{-}E_0$

at $r = a$

$$\frac{\partial\phi}{\partial r} = \sum_{l\neq 1} \left[-(l+1)B_l r^{-(l+2)}\right]_a P_l(\cos\theta)$$

$$+ \left[A_1 - 2B_1 r^{-3}\right]_{r=a} P_1(\cos\theta)$$

ORTHOGONALity of P_ℓ requires this be satisfied term by term.

$B_\ell = 0 \quad \ell \neq 1$

$A_1 = 2 B_1 a^{-3} \qquad B_1 = \frac{A_1}{2} a^3$

$$\boxed{\Phi(r) = -E_0 \cos\theta \left(r + \frac{a^3}{2r^2}\right)}$$

//

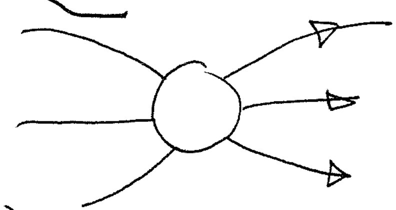
what about inside void

$\phi(r) = +r B\cos\theta$

continuity at ϕ at $r=a$

$B = -E_0 \frac{3}{2}$

Alternate Problem



$\Phi(a) = 0$

$\phi(r) = -E_0 \cos\theta \left(r - \frac{a^3}{r^2}\right)$

$\underset{\sim}{E}(\underset{\sim}{r}) = -\nabla\phi = -\hat{r}\frac{\partial\phi}{\partial r} - \frac{\hat{\theta}}{r}\frac{\partial\phi}{\partial\theta}$

$E_r(r, \theta=0) = +E_0 \cos\theta (1 + 2a^3/r^3)$

note when $r=a$ $\quad E_r = 3E_0$

field is intensified by a factor of 3

Laplaces Eq. in Cylindrical Coordinates

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right)\varphi=0$$

~~Constr~~

Construct a set of basis functions $h(\rho,\varphi,z)$

$$h(\rho,\varphi,z)=R(\rho)\,Q(\varphi)\,Z(z)$$

$$\Rightarrow \quad \frac{\partial^2 Q}{\partial\varphi^2}+\nu^2 Q=0 \quad \Rightarrow \quad Q=e^{\pm i\nu\varphi}$$

when $\varphi\in(0,2\pi]$, $\nu=$ integer

$$\frac{d^2 Z}{dz^2}-k^2 Z=0 \quad \Rightarrow \quad Z=e^{\pm kz}$$

~~Set~~ ~~$x=k\rho$~~

Then

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+\left(k^2-\frac{\nu^2}{\rho^2}\right)R=0$$

Let $k\rho=x$

$$\boxed{\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{\nu^2}{x^2}\right)R=0}$$

Bessel's Eq.

Series Solutions of Bessels Eq

$$R = x^{\alpha} \sum_{j=0}^{\infty} a_j x^j$$

$$(\alpha+j)(\alpha+j-1) a_j x^{\alpha+j} + (\alpha+j) a_j x^{\alpha+j} + a_j x^{\alpha+j+2}$$
$$- \nu^2 a_j x^{\alpha+j} = 0$$

$$a_j\left[(\alpha+j)(\alpha+j\cancel{-1}) + \cancel{(\alpha+j)} - \nu^2\right] = - a_{j-2}$$

$$a_j = - \frac{a_{j-2}}{(\alpha+j)^2 - \nu^2} \qquad \text{as } j \to \infty \quad a_j = \frac{a_{j-2}}{j^2}$$

indicial eq

$$a_0(\alpha^2 - \nu^2) = 0 \qquad j = 0$$
$$\longrightarrow a_1\left[(\alpha+1)^2 - \nu^2\right] = 0 \qquad j = 1$$

not independent.

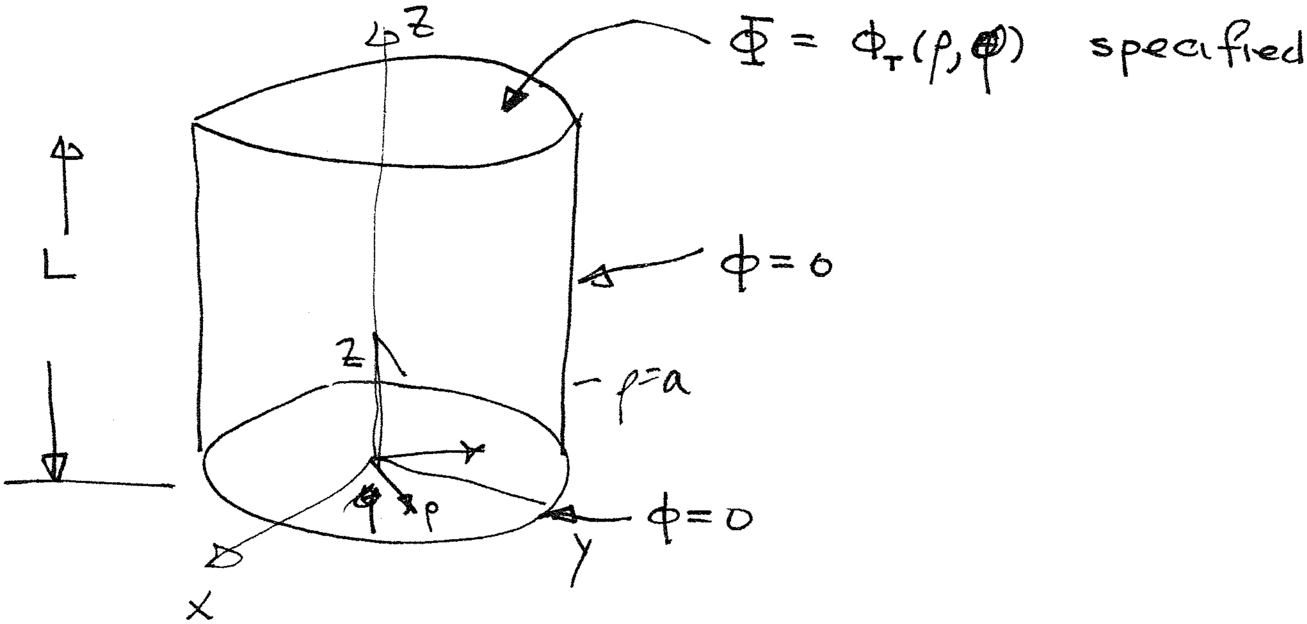
$$\boxed{\alpha = \pm \nu}$$

$$\boxed{\begin{aligned} J_\nu &= \left(\frac{x}{2}\right)^{\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j} \\ J_{-\nu} &= \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j} \end{aligned}}$$

Bessel functions of the first kind.

$\Rightarrow$ Linearly indep. for $\nu \neq$ ~~integer~~ integer.

Find the solution of Laplace's Equation



Solution

$$\phi(\rho, \varphi, z) = \sum_{\nu, k} \left[A_{\nu k} J_\nu(k\rho) + B_{\nu k} N_\nu(k\rho) \right]$$

$$\left[C_{\nu k} \sinh kz + D_{\nu k} \cosh kz \right]$$

$$e^{i\nu\varphi}$$

~~Since Region~~ $0 \le \varphi \le 2\pi$ ϕ must be

periodic in φ ν = integer

since $\rho = 0$ is included $B_{\nu k} = 0$

N_ν is divergent at $\rho = 0$

Since $\phi = 0$ when $z = 0$ $D_{\nu k} = 0$

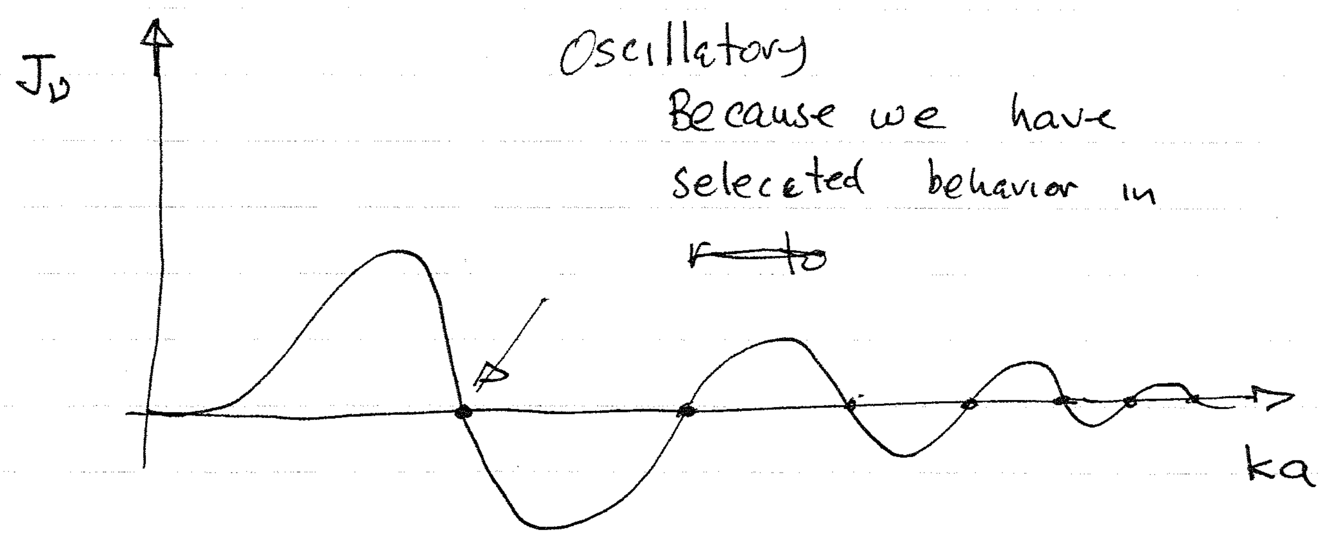
$$\phi(\rho, \varphi, z) = \sum_{k,\nu} A_{\nu k} J_\nu(k\rho) \sinh kz \, e^{i\nu\varphi}$$

How to pick k

Because $\phi(a, \varphi, z) = 0$ pick solutions

such that $J_\nu(ka) = 0$

Oscillatory
Because we have
selecected behavior in
~~r~~ ~~to~~



zero's of Bessel functions

$j_{\nu p}$ = p^{th} zero not countin origin of J_ν

$$J_\nu(j_{\nu p}) = 0$$

$\nu = 0$

~~p=1~~ $p = 1$

~~p=0~~

$$j_{01} = 2.40482\ldots$$

$$j_{02} = 5.52007\ldots$$

$$j_{03} = 8.65372$$

$$\vdots$$

$$j_{0,19} = 58.90698^{39261}$$

$$\Delta \simeq 3.141485$$

$$j_{0,20} = 62.04846919$$

for $\nu = m =$ integer

$$J_{-m} = (-1)^m J_m$$

<u>Neuman Function</u> $\qquad Y_\nu = N_\nu$

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

for $\nu =$ integer

$$N_\nu(x) = \frac{\cos\nu\pi\,\frac{\partial}{\partial\nu}J_\nu(x) - \frac{\partial}{\partial\nu}J_{-\nu}(x)}{\pi\cos\nu\pi}$$

<u>$x \ll 1$</u>

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)}$$

$$N_\nu = \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + .5772\cdots\right] \quad \nu = 0$$

$$- \frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu \quad \nu \neq 0$$

Note that $N_\nu \to \infty$ at $x = 0$ so discard N_ν when the origin is included.

$x \gg 1$

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu(x) = \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

Hankel functions

$$H_\nu^{(1)} = J_\nu + iN_\nu$$

$$H_\nu^{(2)} = J_\nu - iN_\nu$$

If choose $k^2 \Longrightarrow -k^2$

i.e. $Z = e^{\pm ikz}$

then $\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)\right] R = 0$

modified Bessels Eq.

$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$$

$x \ll 1$ $\quad I_\nu \sim x^\nu$

$K_\nu \to \infty$

$x \gg 1$ $\quad I_\nu \sim \frac{e^x}{\sqrt{x}}$

$K_\nu \sim \frac{e^{-x}}{\sqrt{x}}$

Hankel functions

$$H_\nu^1 = J_\nu + iY_\nu \quad \sim Ae^{ix} \quad \text{as } x \to \infty$$

$$H_\nu^2 = J_\nu - iY_\nu \quad \sim e^{-ix} \quad \text{as } x \to \infty$$

---

modified

$$\frac{1}{x}\frac{d}{dx}x\frac{d}{dx}R_\nu - \left(1+\frac{\nu^2}{x^2}\right)R_\nu = 0$$

$$R_\nu = I_\nu(x) \qquad K_\nu(x)$$

$$I_\nu(x) \sim e^x \quad \text{as } x \to \infty$$
$$K_\nu(x) \sim e^{-x} \quad \text{as } x \to \infty$$

Like cosh and sinh

Thus, for each value of ν we sum over all k such that

$$J_\nu(ka) = 0$$

$$k_{\nu p} = \frac{j_{\nu p}}{a}$$

$$\phi(\rho, \varphi, z) = \sum_{\nu=-\infty}^{\infty} \sum_{p=1}^{\infty} A_{\nu p} J_\nu(k_{\nu p} r) e^{i\nu\varphi} \sinh k_{\nu p} z$$

$$J_\nu(k_{\nu p} r) \cancel{J_\nu(k_{\nu p'} r)}$$

orthogonal set

$$\int_0^a r\,dr\, J_\nu(k_{\nu p} r) J_\nu(k_{\nu p'} r) = 0 \qquad p \neq p'$$

Bessel's Equation

$$\frac{1}{x}\frac{d}{dx}x\frac{d}{dx}R_\nu + \left(1-\frac{\nu^2}{x^2}\right)R_\nu = 0$$

ORDINARY Bessel functions

$R_\nu = \quad J_\nu(x) \qquad J_{-\nu}(x) \qquad (\nu \neq$ integer

$$J_\nu(x) = \left(\tfrac{1}{2}x\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^k}{k!\,\Gamma(\nu+k+1)}$$

$(\nu+k)!$

$$Y_\nu(x) \text{ or } N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

for ν an integer n take limit $\nu \to n$

for small x $\quad J_\nu \sim \left(\tfrac{1}{2}x\right)^\nu \dfrac{1}{\nu!}$

$$J_{-\nu} \sim \left(\tfrac{1}{2}x\right)^{-\nu} \frac{1}{\Gamma(k+1-\nu)}$$

could be negative

$$z\,\Gamma(z) = \Gamma(z+1) \qquad \boxed{\Gamma(z-1) = \frac{\Gamma(z)}{(z-1)}}$$

$$A_{\nu p} = \frac{\int_0^a \int_0^{2\pi} d\varphi \, \rho d\rho \; e^{-i\nu\varphi} J_\nu(k_{\nu p}\rho) \, \phi_{TOP}(\rho,\varphi)}{2\pi \sinh k_{\nu p} \underbrace{\int_0^a \rho \, d\rho \; J_\nu^2(k_{\nu p}\rho)}_{\frac{a^2}{2} J_\nu'^2(k_{\nu p} a)}}$$

Alternative Proof

In 3D spherical

$$\delta(\hat{\underline{r}}-\underline{r}') = \frac{\delta(\cos\theta-\cos\theta')\,\delta(r-r')\,\delta(\varphi-\varphi')}{D}$$

$$\int \underbrace{r^2 dr\, d\cos\theta\, d\varphi}_{dV}\ \delta(\underline{r}-\underline{r}') = 1$$

$$\int r^2 dr\, d\cos\theta\, d\varphi\ \frac{\delta(\cos\theta-\cos\theta')\,\delta(r-r')\,\delta(\varphi-\varphi')}{D} = \frac{r^2}{D} = 1$$

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}G_{\ell m} - \frac{\ell(\ell+1)}{r^2}G_{\ell m} = -\frac{4\pi}{r^2}\int Y_{\ell m}^{*}\, d\phi\, \cancel{\cos}\, d\cos\theta\ \delta(\cos\theta-\cos\theta')\,\delta(\varphi-\phi')\,\delta(r-r')$$

$$=$$

$$G = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\underbrace{\left[A_{\ell m}r^{\ell} + B_{\ell m}r^{-(\ell+1)}\right]}_{G_{\ell m}}Y_{\ell m}(\theta,\phi)$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}G_{\ell m} - \frac{\ell(\ell+1)}{r^2}G_{\ell m} = -\frac{4\pi}{r^2}Y_{\ell m}^{*}(\theta',\phi')\,\delta(r-r')}$$

skip!

for $r > r'$

$$G_{\ell m} = B_{\ell m}\, r^{-(\ell+1)}$$

for $r < r'$

$$G_{\ell m} = A_{\ell m}\, r^{\ell}$$

continuity at $r = r'$ $\qquad A_{\ell m}\, r'^{\ell} = B_{\ell m}\, r'^{-(\ell+1)}$

$$\int_{r'-\epsilon}^{r'+\epsilon} \left[\frac{d}{dr} r^2 \frac{d}{dr} G_{\ell m} - \ell(\ell+1) G_{\ell m} \right] = -4\pi\, Y^*_{\ell m}(\theta', \phi')$$

$$r'^2 \left[\frac{-(\ell+1) B_{\ell m}}{r'^{(\ell+2)}} - \ell A_{\ell m}\, r'^{(\ell-1)} \right] = -4\pi\, Y^*_{\ell m}(\theta', \phi')$$

$$B_{\ell m} = A_{\ell m}\, r'^{(2\ell+1)}$$

$$-\frac{(\ell+1)}{r'^{\ell}} A_{\ell m}\, r'^{(2\ell+1)} - \ell A_{\ell m}\, r'^{(\ell+1)} = -4\pi\, Y^*_{\ell m}(\theta', \phi')$$

$$A_{\ell m} = \frac{4\pi}{-(2\ell+1)\, r'^{(\ell+1)}}\, Y^*_{\ell m}$$

$$B_{\ell m} = \frac{4\pi}{(2\ell+1)} r'^{\ell} Y_{\ell m}^{*}$$

$$\boxed{G = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{(2\ell+1)} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}^{*}(\theta', \varphi') Y_{\ell m}(\theta, \phi)}$$

$$= \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

THUS

$$\boxed{P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^{*}(\theta', \varphi') Y_{\ell m}(\theta, \varphi)}$$

$$\int_0^a \rho d\rho \, J_\nu^2(k\rho) = \frac{a^2}{2}\left[J_\nu'^2(ka) + \left(1 + \frac{\nu^2}{k^2a^2}\right) J_\nu^2(ka)\right]$$

if k choosen such that $J_\nu(ka) = 0$

VIZ $k = \frac{j_{\nu p}}{a}$

THEN
$$\int_0^a \rho d\rho \, J_\nu^2\left(\frac{\rho}{a} j_{\nu p}\right) = \frac{a^2}{2} J_\nu'^2(j_{\nu p})$$

~~THUS~~ ~~WI~~

coefficients $A_{\nu p}$ choosen so

that at $z = L$

$$\phi_{TOP}(\rho, \varphi) = \sum_{\nu=-\infty}^{\infty} \sum_{p=1}^{\infty} A_{\nu p} J_\nu(k_{\nu p}\rho) e^{i\nu\varphi} \sinh k_{\nu p} L$$

Chapter 9

~~XXX:~~ ~~Section 4~~ MICROSCOPICS

Multipoles and Multipole expansions

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} \quad \text{expand for } |\mathbf{x}'| << \mathbf{x}$$

$$\cong \frac{1}{|x|} - \mathbf{x}'\cdot\nabla\frac{1}{|\mathbf{x}|} + \frac{1}{2}\mathbf{x}'\mathbf{x}':\nabla\nabla\frac{1}{|\mathbf{x}|} + \cdots$$

$$\nabla\frac{1}{|\mathbf{x}|} = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$$

$$x'x':\nabla\nabla\frac{1}{|\mathbf{x}|} = \sum_{i,j} x_i' x_j' \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{1}{|\mathbf{x}|}$$

$$= -\sum_{i,j} x_i' x_j' \frac{\partial}{\partial x_i}\frac{x_j}{|\mathbf{x}|^3}$$

$$= -\sum_{i,j} x_i' x_j' \left[\frac{\delta_{ij}}{|\mathbf{x}|^3} - 3\frac{x_i x_j}{|x|^5}\right]$$

$$\phi(x) = \int d^3x' \frac{\rho(x')}{4\pi\epsilon_0}\left[\frac{1}{|x|} + \frac{x'\cdot x}{|x|^3} + \frac{1}{2}\sum_{i,j}\frac{x_i'x_j'}{|x|^5}\left[3x_ix_j - |x|^2\delta_{ij}\right]\right]$$

$$= \frac{1}{2}\left[\sum_{i,j}\frac{x_i'x_j'}{|x|^5}3x_ix_j - \frac{|x'|^2|x|^2}{|x|^5}\right]$$

$$= \frac{1}{2}\sum_{i,j}\left[\frac{x_ix_j}{|x|^5}\left[3x_i'x_j' - |x'|^2\delta_{ij}\right]\right]$$

THUS,

$$\phi(x) = \frac{q}{4\pi\epsilon_0|x|}\cancel{\int d^3x'\rho(x')} + \frac{p\cdot x}{4\pi\epsilon_0|x|^3} + \frac{1}{2}\sum_{ij}\frac{Q_{ij}}{4\pi\epsilon_0}\frac{x_ix_j}{|x|^5}$$

$q = \int d^3x'\rho(x')$ monopole moment 1 #

$p = \int d^3x'\, x'\rho(x')$ dipole moment 3 #'s $\ell=1$ $m=\pm 1,0$

note Q_{ij} is inde

$$Q_{ij} = \int d^3x'\rho(x')\left[3x_i'x_j' - |x'|^2\delta_{ij}\right]$$ #'s

$\boxed{\text{Compare}}$ sizes $\ell=2$ $m=\pm 2,\pm 1,0$ ← 5 indep#!

Quadrupole moment $\boxed{\text{Traceless}}$

$\cancel{3 + \frac{\ell^2-\ell}{2}}$ $9-3$ $= 5$ #S

$\cancel{\ell+\ell^2}$!

$\sum_i Q_{ii} = 0$

lowest non zero moment independ of origin of C.S.

8.

Alternate expansion

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi\, Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)}{(2l+1)} \frac{r'^{l}}{r^{l+1}}$$

$\rho \neq 0$ for $r' < R$

assume $r > R$

$|x'| = r' < r = |\mathbf{x}|$

$$\phi(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{\cancel{4\pi}\, q_{lm}/\epsilon_0}{(2l+1)\, r^{l+1}} Y_{lm}(\theta,\phi)$$

where $$q_{lm} = \int d^3x'\, r'^{l}\, Y^*_{lm}(\theta',\varphi')\,\rho(\mathbf{x})$$

$\boxed{\text{multipole moments}}$

$$\sum_i Q_{ii} = \sum_i \left[3x_i^2 - |x|^2\delta_{ii}\right]$$

$$3(x_1^2+x_2^2+x_3^2) - 3|x|^2 = 0$$

Dipole Field

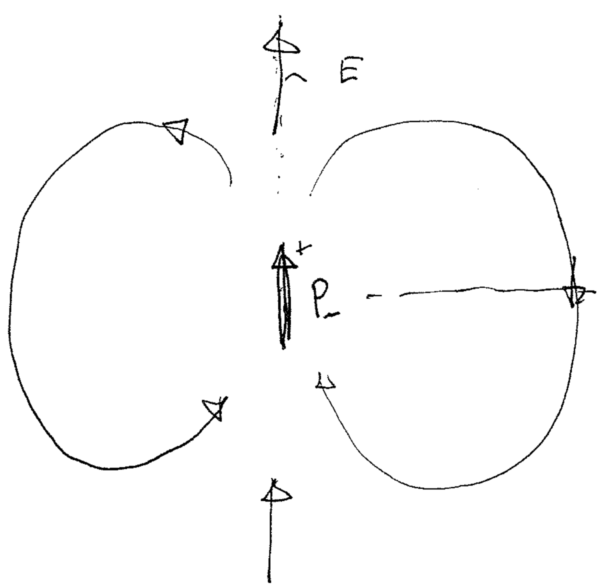
$$\phi(x) = \frac{p\cdot x}{4\pi\epsilon_0 |x|^3}$$

falls like $\frac{1}{x^3}$

aligned with p along axis

anti parallel in plane $\perp$ to p

$$E = -\nabla\phi = -\frac{1}{4\pi\epsilon_0}\left[\frac{p}{|x|^3} - \frac{3x\, p\cdot x}{|x|^5}\right]$$



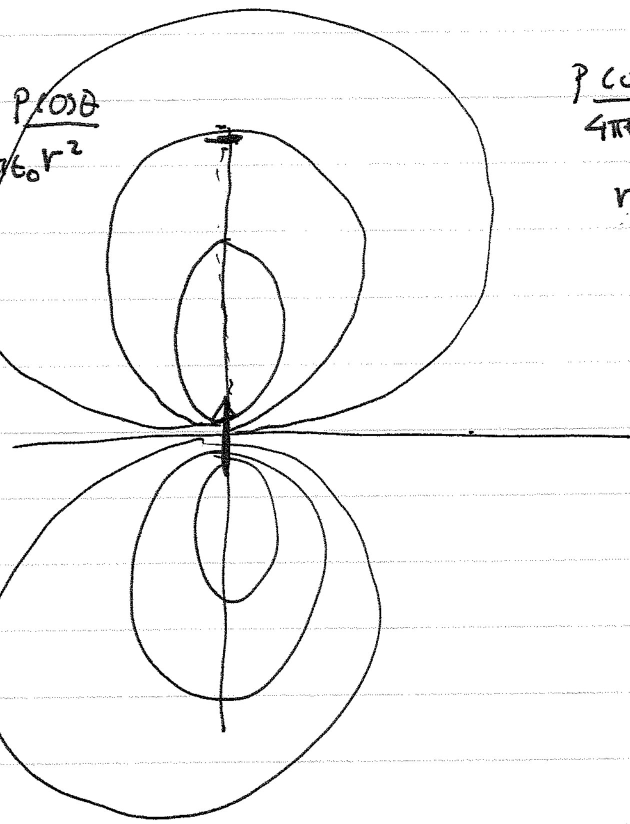
if $x \parallel p$

$$|E| = \frac{\overbrace{(3-1)}^{2}|p|}{4\pi\epsilon_0 |x|^3}$$

if $x \perp p$

$$E = -\frac{p}{4\pi\epsilon_0 |x|^3}$$

$$\phi = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$



$$\frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \text{const}$$

$r^2 \propto \cos\theta$

$\theta = \pi/2$

$$\nabla^2 \phi_{ext} = -\cancel{4\pi}\, \rho_{ext}/\epsilon_0$$

$$\nabla^2 \phi = -\cancel{4\pi}\, \rho/\epsilon$$

$$\phi_T = \phi + \phi_{ext}$$

$$W_T = \frac{1}{2}\int d^3x \left[\rho_{ext}\phi_{ext} + \rho_{ext}\phi + \rho\,\phi_{ext} + \rho\,\Phi \right]$$

$$\int d^3x\, \rho_{ext}\Phi = \int d^3x \int d^3x' \frac{\rho_{ext}(x)\rho(x')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

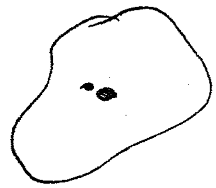
$$= \int d^3x\, \rho\, \Phi_{ext}$$

work required to assemble ext charge

work required to assemble internal

$$W_T = \frac{1}{2}\int d^3x \left[\rho_{ext}\phi_{ext} + 2\rho\,\phi_{ext} + \rho\,\Phi \right]$$

work required to assemble ...



expand $\phi_{ext}(\underset{\sim}{x})$ about $\underset{\sim}{x}=0$ origin of coordinate system for which we have calculated moments of $\rho(\underset{\sim}{x})$

$$\Phi_{ext} \cong \phi_0 + \underset{\sim}{x} \cdot \nabla\phi + \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}\phi$$

$$= \phi_0 - \underset{\sim}{x} \cdot \underset{\sim}{E} - \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial E_i}{\partial x_j}$$

note since $\nabla \cdot E = \sum_j \frac{\partial E_j}{\partial x_j} = 0$ for external field

$$\phi_{ext} \cong \phi_0 - \underset{\sim}{x} \cdot \underset{\sim}{E} - \frac{1}{2} \sum_{ij} x_i x_j \frac{\partial E_i}{\partial x_j}$$

add 0

$$- \frac{1}{2} \left[\sum_{ij} x_i x_j \frac{\partial E_i}{\partial x_j} - \sum_{ij} \frac{x_i x_j \delta_{ij}}{3} \underbrace{\sum_{\ell k} \frac{\partial E_\ell}{\partial x_k} \delta_{\ell k}}_{\text{zero}} \right]$$

change subscripts.

$$- \frac{1}{2} \left[\sum_{ij} x_i x_j \frac{\partial E_i}{\partial x_j} - \underbrace{\sum_{\ell k} \frac{x_\ell x_k \delta_{\ell k}}{3}}_{\frac{\underset{\sim}{x}^2}{3}} \sum_{ij} \frac{\partial E_i}{\partial x_j} \delta_{ij} \right]$$

$$- \frac{1}{2} \sum_{ij} \frac{\partial E_i}{\partial x_j} \left[x_i x_j - \frac{|x|^2}{3} \delta_{ij} \right]$$

$$\Phi_{ext} \cong \phi_0 - \underset{\sim}{x} \cdot \underset{\sim}{E} - \frac{1}{2} \sum_{ij} x_i x_j \left[\cancel{\frac{1}{2}\left(\frac{\partial E_i}{\partial x_j} + \frac{\partial E_j}{\partial x_i}\right)} \frac{\partial E_i}{\partial x_j} - \delta_{ij} \frac{1}{3} \left(\frac{\partial E_i}{\partial x_j}\right) \right]$$

$$W = \int d^3x \, \rho(\underset{\sim}{x}) \, \Phi_{ext} = q \, \phi(0) - \underset{\sim}{p} \cdot \underset{\sim}{E} - \frac{1}{6} \sum_{ij} Q_{ij} \frac{\partial E_i}{\partial x_j}$$

supose that charge distribution was ~~locate~~ centered at $\underset{\sim}{x}_0$

Higher order in $\left(\frac{a}{L}\right)$

$$W = q \, \phi(\underset{\sim}{x}_0) - \underset{\sim}{p} \cdot \underset{\sim}{E}(\underset{\sim}{x}_0) - \frac{1}{6} \sum_{ij} Q_{ij} \left.\frac{\partial E_i}{\partial x_j}\right|_{\underset{\sim}{x}_0}$$

a = size of object
L = scale length of external field

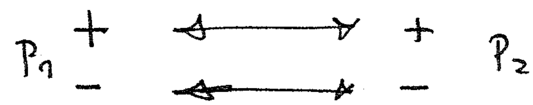
FORCE on charge distribution can be obtained by method of virtual displacemts

$F = -\delta W$

$$\mathbf{p}_2 \cdot \hat{\mathbf{r}} = \mathbf{p}_1 \cdot \hat{r} = 0$$

$$W = \frac{\mathbf{p}_2 \cdot \mathbf{p}_1}{|r|^3}$$

Repulsive if alligned



attractive if

p_1 exerts torque on p_2

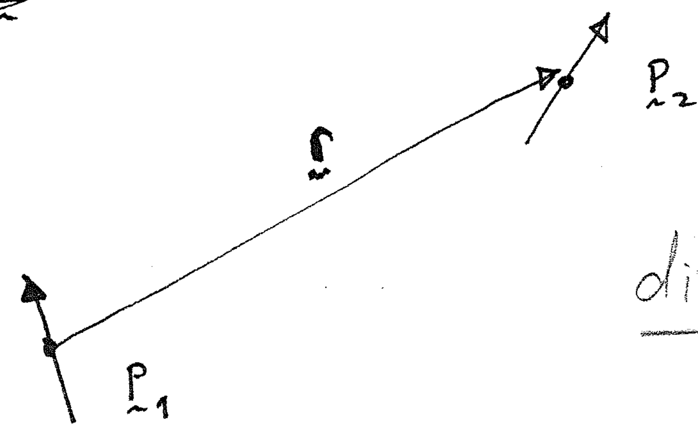
---

want to be anti alligned if $\vec{p}_1 \cdot \hat{r} = \vec{p}_2 \cdot \hat{r} = 0$

~~Example suppose $Q_{ij} = 0$~~ **SKIP**

Interaction of two permanent dipoles

$\cancel{\mathbf{E}} =$



dipole field

$$\phi_1 = \frac{\mathbf{p}_1 \cdot \mathbf{r}}{4\pi\epsilon_0 |\mathbf{r}|^3} \qquad E_1 = -\nabla \phi_1$$

$$\mathbf{E}_1 = -\frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{p}_1}{|\mathbf{r}|^3} - \frac{3\,\mathbf{r}\,\mathbf{r}\cdot\mathbf{p}_1}{|\mathbf{r}|^5}\right]$$

unit vector from 1 to 2 or vice versa

$$W = -\mathbf{p}_2 \cdot \mathbf{E}_1 = \frac{\mathbf{p}_2\cdot\mathbf{p}_1 - 3(\mathbf{p}_2\cdot\hat{\mathbf{r}})(\mathbf{p}_1\cdot\hat{\mathbf{r}})}{4\pi\epsilon_0 |\mathbf{r}|^3}$$

may be attractive or repulsive depending on orientation

$\mathbf{p}_2\cdot\mathbf{p}_1 - 3(\mathbf{p}_2\cdot\mathbf{r})(\mathbf{p}_1\cdot\mathbf{r}) > 0$

4.3

Ponderable Material

Dielectric Material

The imposition of an electric field on the material induces a rearrangement of the charge in the material.

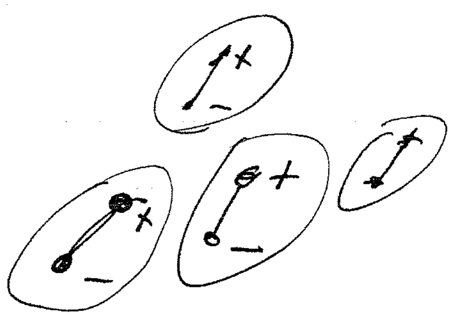
~~Two~~ Treatment of dielectric material results in the introduction of the dielectric constant ϵ (The simplest case)

Two steps are required in the treatment of dielectric material

1) replacing point charges associated with individual atoms by continuous charge distributions

2) expressing the induced charge as a function of the electric field.

Consider a model of the media as an ensemble of a large number of dipoles of ~~str~~ varying strength



let $N(\underline{x})$ be the local number density of dipoles (analogous to a charge density)

Macroscopic Treatment

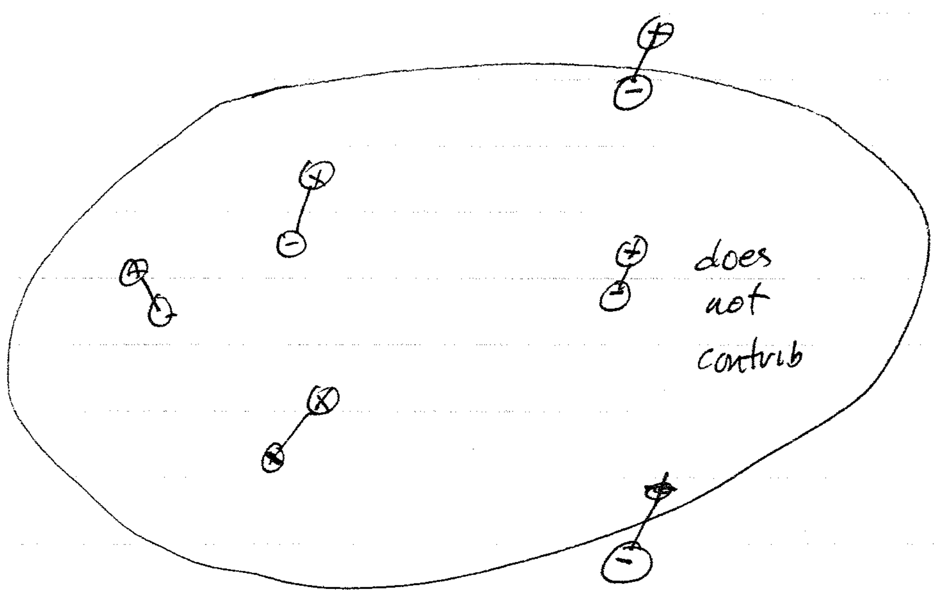
~~however,~~
each dipole has no net charge

but a dipole moment $\underset{\sim}{p}(\underset{\sim}{x})$

$\underset{\sim}{p}$ = dipole moment of dipole centered at $\underset{\sim}{x}$

Because dipoles have + and −

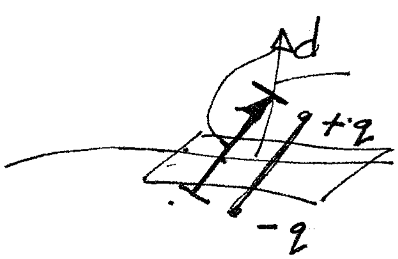
charges slightly displaced there can be a local charge density
(induced charge density)
due to non uniformity of $\underset{\sim}{p}$ or N

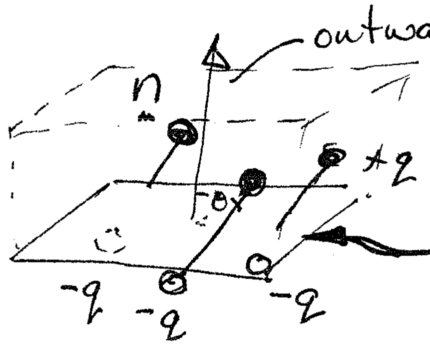


what is the total umbalanced charge inside this volume?

Any dipole entirely inside the volume will not contribute to the net charge, ~~only that~~

Only those straddling the surface will contribute





small element of area on surface

~~# of negative~~ amount charges inside

$$= -q\, dA\, d\cos\theta\, N$$

separation

density

~~local~~ angle between $\underset{\sim}{p}$ and $\underset{\sim}{n}$

$\underset{\sim}{p} =$ $\quad |\underset{\sim}{p}_m| = qd$

$$Q_d = -\int_S dA\, \cancel{\hat{n}}\, \hat{n}\cdot \underbrace{N\underset{\sim}{p}}$$

$$= -\int_S dA\, \underset{\sim}{n}\cdot \underset{\sim}{P}$$

~~densit~~

<u>electric polarization</u> $\underset{\sim}{P} = N\underset{\sim}{p}$ # density – dipole moment

~~dipole~~ induced charge density

$$Q_d = \int d^3x\, \rho_d = -\int_S dA\, \underset{\sim}{n}\cdot \underset{\sim}{P}$$

divergence theorem

$$= -\int d^3x\, \nabla\cdot \underset{\sim}{P}$$

induced charge density

$$\rho_d = -\nabla\cdot \underset{\sim}{P}$$

total charge density

free charge density

$$\rho_{free} - \underbrace{\nabla\cdot \underset{\sim}{P}}$$

induced charge density

TOTAL charge density

$$\nabla\cdot \overset{\epsilon_0}{\mathbf{E}} = +\cancel{4\pi}\rho_T$$

free, induced

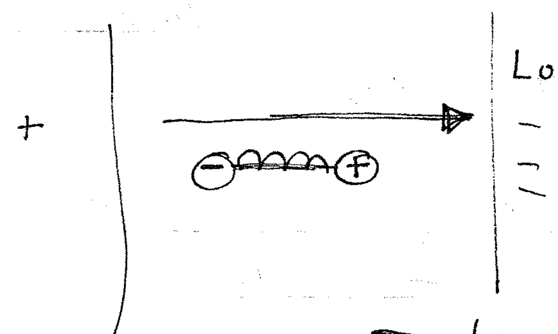
$$= +\cancel{4\pi}\left[\rho - \nabla\cdot\mathbf{P}\right]$$

$$\nabla\cdot(\underbrace{\epsilon_0\mathbf{E} + \cancel{4\pi}\mathbf{P}}) = 4\pi\underline{\rho_{free}}$$

call this $\mathbf{D}$ <u>electric displacement</u>

Part 2   How is $\mathbf{P}$ determined

<u>Constituative relation</u>



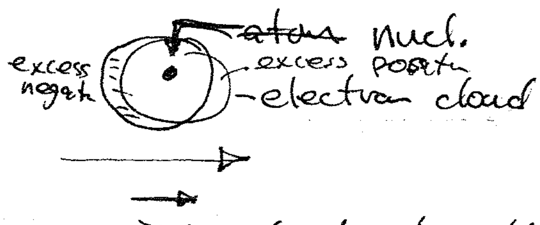
Local electric field

For a ~~lo~~ low density

$$\overset{\text{dipole moment}}{\mathbf{p}} = \gamma\epsilon_0\mathbf{E}$$

number density

$$\mathbf{P} = N\mathbf{p} = N\gamma\epsilon_0\mathbf{E}$$



induced dipole moment

electric susceptibility

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$$

what does

$$\underset{\sim}{P} = \epsilon_0 \chi_e \underset{\sim}{E} \quad \text{assume!}$$

1) $\underset{\sim}{P}$ is parallel to $\underset{\sim}{E}$ medium is isotropic. That is, there is no prefered direction in the medium and thus $\underset{\sim}{P}$ will allign itself with $\underset{\sim}{E}$

counter examples

1) crystals
2) plasmas in magnetic fields

$$\underset{\sim}{P} = \epsilon_0 \underset{\approx}{\chi} \cdot \underset{\sim}{E} \qquad \text{susceptibility tensor}$$

2) if χ_e is a constant, ~~the~~ independent of $\underset{\sim}{E}$ then the then it is assumed that the medium is linear valid for $\underset{\sim}{E}$ small enough

counter examples

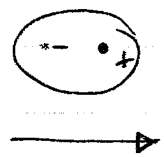
any medium ~~has its~~ has its limit

$$\chi(|E|^2)$$

eg Kerr effect in solids

effect is strong in optical fibers

3) Response is instantaneous in time



$\underset{\sim}{P}$

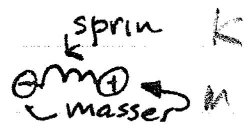
$$\underset{\sim}{P} = 4\pi\chi \underset{\sim}{E}$$

$\underline{E}$

assumes that if I change $\underset{\sim}{E}$ $\underset{\sim}{P}$ instantly adjusts itself

~~for th.~~ ~~assum~~

<u>counter examples</u>: any medium if ~~almost~~ E changes fast enough



$\longleftrightarrow$ natural frequency of vibration

ω_r $\qquad \omega_{res} = \sqrt{\frac{K}{m}}$

if $\left| \frac{1}{E}\frac{\partial E}{\partial t} \right| \ll \omega_r$ then instantaneous

otherwise

$$P(t) = \int_{-\infty}^{t} dt' \, E(t') \, K(t-t')$$

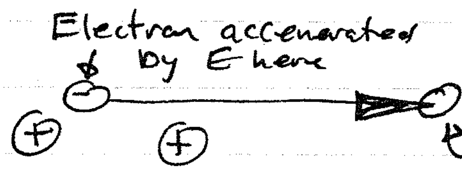
4) Dielectric constant is local in space

$\underset{\sim}{P}(x)$ depends only on $\underset{\sim}{E}(\underset{\sim}{x})$

counter example

$$\underset{\sim}{P}(\underset{\sim}{x}) = \int d^3x' \, \overset{\epsilon_0}{\cancel{4\pi}} \, \chi(\underset{\sim}{x}, x') \, \underset{\sim}{E}(x')$$

nonlocal dielectric plasma



← charge imbalance over here

$$\nabla\cdot\overbrace{(\epsilon_0+\epsilon_0\chi_e)}^{\text{dielectric constant}}\underset{\sim}{E} = \cancel{4\pi}\rho$$

$$\epsilon = \epsilon_0(1+\chi_e) \qquad \text{vacuum } \epsilon = 1$$

also written

$$\nabla\cdot\underset{\sim}{D} = \cancel{4\pi}\rho$$

~~$D = \epsilon E$~~ Remember there is a long list of assumptions implied by writim this formula

$$\underset{\sim}{D} = \epsilon_0\underset{\sim}{E} + \cancel{4\pi}\underset{\sim}{P} = \epsilon\underset{\sim}{E}$$

$\boxed{\text{D can not be measured directly, it is not a fundamental quantity like E}}$

$10+4=14$

Electrostatics with Dielectrics

$$\nabla \times \underset{\sim}{E} = 0$$

$$\nabla \cdot \underset{\sim}{D} = \cancel{4\pi}\rho_f$$

$$\underset{\sim}{D} = \epsilon(\underset{\sim}{x})\,\underset{\sim}{E}$$

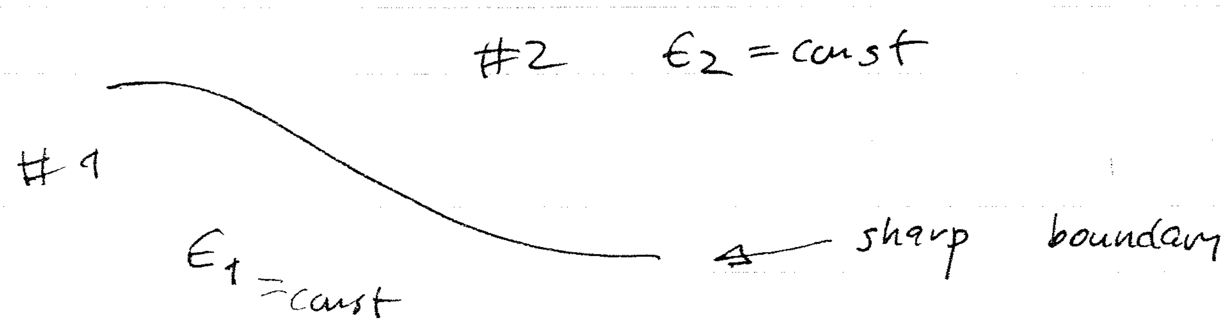
dielectric constant depends on space

$\nabla \times \underset{\sim}{E} = 0$ implies $\underset{\sim}{E} = -\nabla\phi$

$$\nabla \cdot \epsilon(\underset{\sim}{x})\,\nabla\Phi = -4\pi\rho_f$$

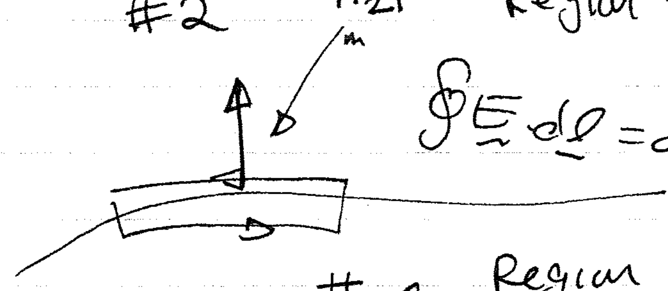
"Poisson like" Equation determines ϕ

most frequent types of problem



Boundary Conditions

Region #2, ϵ_2



$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = 0$$

$$\nabla \times \underset{\sim}{E} = 0.$$

Region #1, ϵ_1

$$\underset{\sim}{E}_{t_1} - \underset{\sim}{E}_{t_2} = 0$$

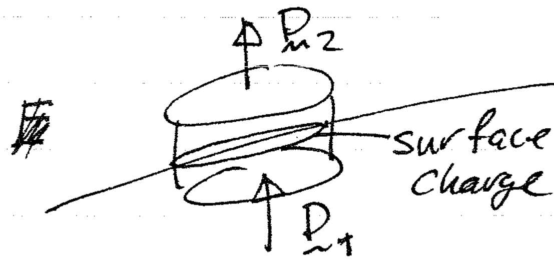
$\underset{\sim}{n}_{21}$ = normal from #1 to #2

$$(\underset{\sim}{E}_1 - \underset{\sim}{E}_2) \times \underset{\sim}{n}_{21} = 0$$

$$\nabla \cdot \underset{\sim}{D} = \cancel{4\pi} \rho_{free}$$

free charge

$$\int_S da\, \underset{\sim}{n} \cdot \underset{\sim}{D} = \cancel{4\pi}\, q_{enclosed}$$



$$(\underset{\sim}{D}_2 - \underset{\sim}{D}_1) \cdot \underset{\sim}{n}_{21} = \cancel{4\pi}\, \sigma_f$$

"free" surface charge

"free surface charge is the surface charge you put on the surface"
not the surface charge induced by the discontinuity of the dielectric constant

In most cases $\sigma_f = 0$

$$(\underset{\sim}{D}_2 - \underset{\sim}{D}_1) \cdot \underset{\sim}{n}_{21} = 0$$

writing

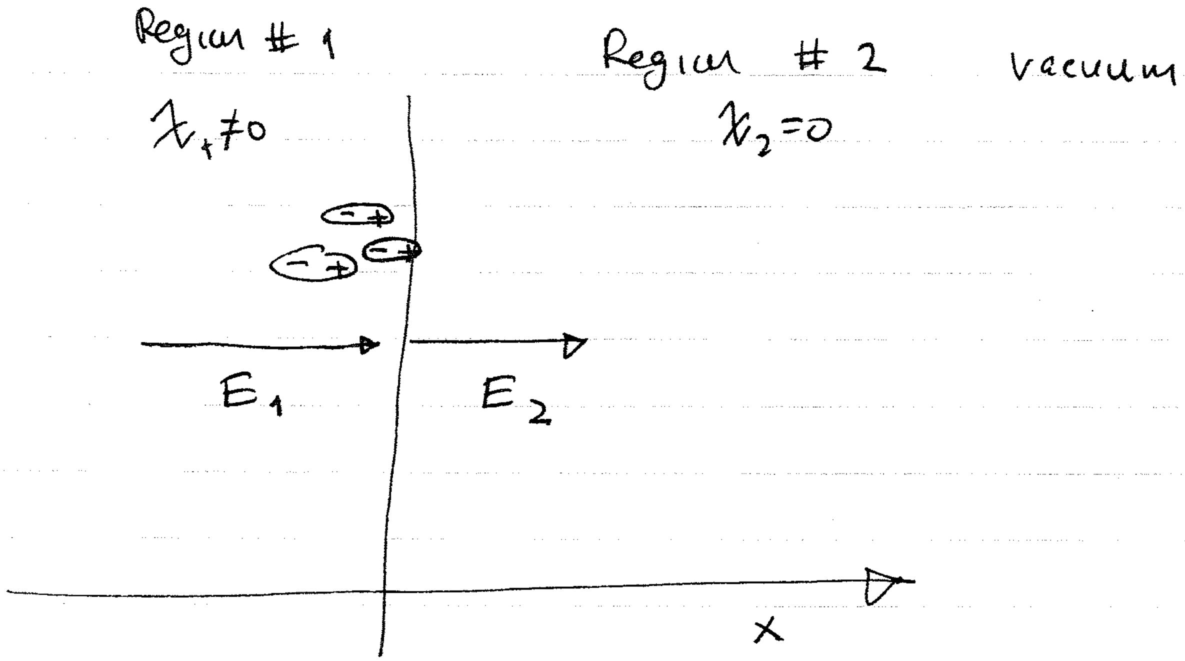
$$\underset{\sim}{D}_2 = \epsilon_2 \underset{\sim}{E}_2 \qquad \epsilon_2 = \epsilon_0 + 4\pi\chi_2$$

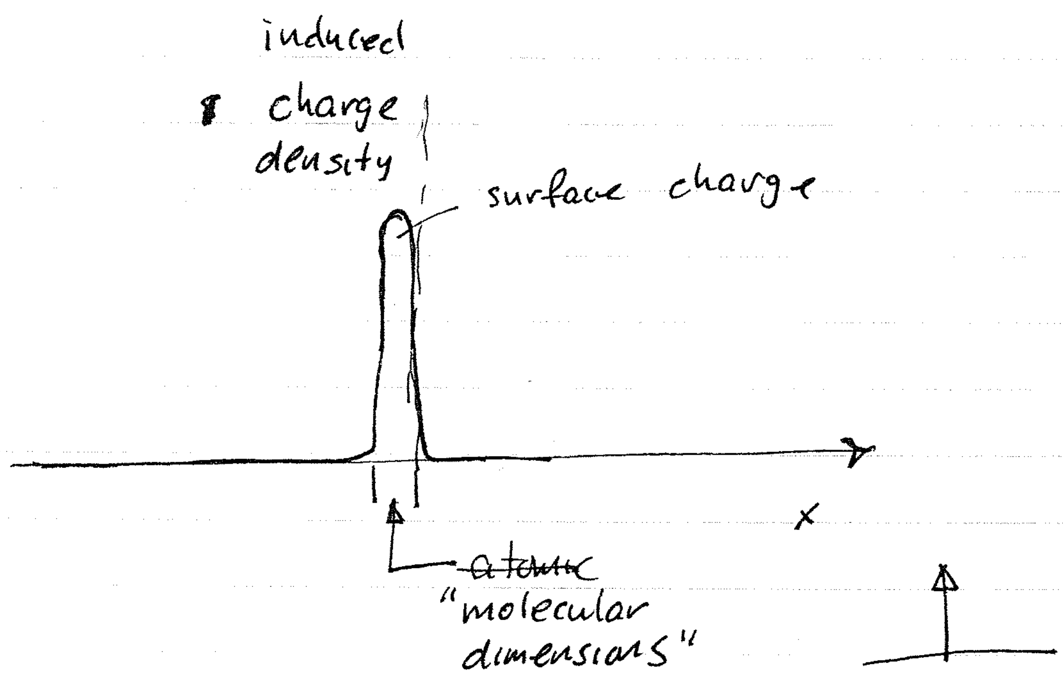
$$\underset{\sim}{D}_1 = \epsilon_1 \underset{\sim}{E}_1 \qquad \epsilon_1 = \epsilon_0 + 4\pi\chi_1$$

$$\left[\epsilon_2 \underset{\sim}{E}_2 - \epsilon_1 \underset{\sim}{E}_1\right] \cdot \underset{\sim}{n}_{21} = 0$$

$$\epsilon_0 (\underset{\sim}{E}_2 - \underset{\sim}{E}_1) \cdot n_{21} = + \cancel{4\pi} \underbrace{\left[-(\chi_2 \underset{\sim}{E}_2 - \chi_1 E_1) \cdot n_{21}\right]}_{\sigma_i}$$

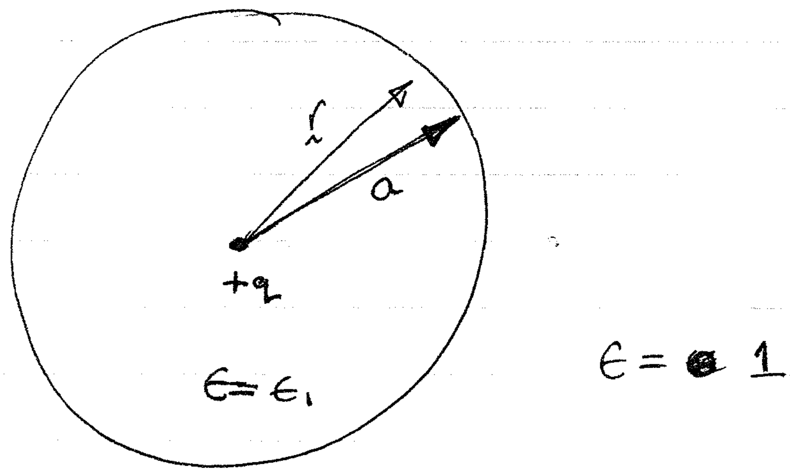
induced surface charge





Continuity of $\underset{\sim}{D}$ accounts for induced surface charge.

Consider a free point charge $+q$ at the center of an uncharged dielectric sphere



Outside

$$\nabla^2 \phi = 0$$

solution — total charge inside a **sphere**

$$\phi = \frac{Q}{4\pi\epsilon_0 |\mathbf{r}|}$$

general solution with spherical symmetry

Alternatively

inside

$$-\epsilon \nabla^2 \phi = \cancel{4\pi} q\, \delta(\mathbf{r})$$

point charge at origin

outside:

$$\nabla^2 \phi = 0 \qquad BC \quad \phi \to 0 \quad \text{as } r \to \infty$$

* Solution

$$\phi = \frac{Q}{4\pi\epsilon_0 |\underset{\sim}{r}|}$$

$Q \sim$ TBD
(However we know
$Q = q$)

inside: $\quad \nabla \cdot \underset{\sim}{D} = q\,\delta(r)$ $\qquad \underset{\sim}{D} = \epsilon \underset{\sim}{E}$

$\underset{\sim}{E} = -\nabla\phi$

$$-\epsilon \nabla^2 \phi = q\,\delta(r)$$

Solution: $\quad \phi = \dfrac{q}{4\pi\epsilon |r|} + C$

BC's at $r = a$

Continuity of D_r

$$\epsilon_0 \frac{\partial \phi}{\partial r}\bigg|_{r=a} = \epsilon \frac{\partial \phi}{\partial r}\bigg|_{r=a}$$

outside $\qquad$ inside

$$-\frac{Q}{4\pi a^2} = -\frac{q}{4\pi a^2}$$

$\cancel{q = Q}$ $\quad Q = q$

continuity of ϕ

$$\frac{q}{4\pi\epsilon a} + C = \frac{q}{4\pi\epsilon_0 a}$$

$$C = \frac{q}{4\pi a}\left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon}\right)$$

solution,

const.

$$\phi = \frac{q}{4\pi\epsilon|r|} + \phi_0$$

apply boundary conditions at $r=a$

just outside just inside

$$-\left[\epsilon_0 \frac{\partial\phi}{\partial r}\Big|_{r=a+0} - \epsilon\frac{\partial\phi}{\partial r}\Big|_{r=a-0}\right] = 0$$

no free charge at surface

$$\epsilon_0 \frac{Q}{a^2} - \frac{q}{a^2} = 0$$

~~$q = Q$~~

$Q = q$

continuity of ϕ

OUT in

$$\frac{q}{4\pi\epsilon_0 a} = \frac{q}{4\pi\epsilon a} + \phi_0$$

$\phi(r)$

$$\phi_0 = \frac{q}{4\pi a}\left(1 - \frac{\epsilon_0}{\epsilon}\right)$$

what is the induced charge density?

use $\epsilon_0 \int_s da\, \underset{\sim}{\hat{n}} \cdot \underset{\sim}{E} = q_{total}$

inside

$$E_r = -\frac{\partial \phi}{\partial r} = \frac{q}{4\pi \epsilon r^2}$$

outside

$$E_r = -\frac{\partial \phi}{\partial r} = \frac{q}{4\pi\epsilon_0 r^2}$$

Inside

$$4\pi r^2 \epsilon_0 \frac{q}{4\pi \epsilon r^2} = \cancel{4\pi}\, q_{total}$$

$$q_{total} = \epsilon_0 \frac{q}{\epsilon} = q + q_{induced}$$

free charge

induced charge

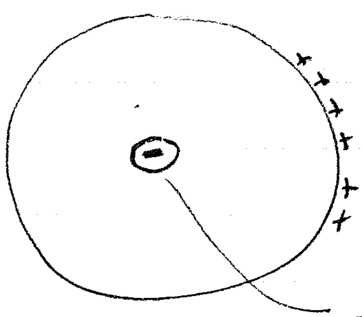
induced charge $= -q\left(1 - \frac{\epsilon_0}{\epsilon}\right)$

<u>outside</u>

$$4\pi r^2 \epsilon_0 \frac{q}{4\pi\epsilon_0 r^2} = \cancel{4\pi}\, q_{total}$$

total charge $= q$

induced charge density



surface charge $\sigma = \dfrac{q\left(1-\frac{\epsilon_0}{\epsilon}\right)}{4\pi a^2}$

AREA

point charge $-q\left(1-\frac{\epsilon_0}{\epsilon}\right)$