Chapter 15

NORMALIZation

$$\int_{-1}^{1} dx\, P_{\ell'}^{m}(x)\, P_{\ell}^{m}(x) = \frac{2}{2\ell+1}\, \frac{(\ell+m)!}{(\ell-m)!}\, \delta_{\ell\ell'}$$

if $m=0$

$$\int_{-1}^{1} P_\ell(x)\, P_\ell(x')\, dx = \frac{2}{2\ell+1}\, \delta_{\ell\ell'}$$

THAT IS

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \, Y_{l'm'}^{*} Y_{lm} = \delta_{ll'} \delta_{mm'}$$

Completeness Relation

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}^{*}(\theta',\varphi') Y_{lm}(\theta,\varphi) = \delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta')$$

$$= \frac{\delta(\varphi-\varphi')\delta(\theta-\theta')}{|\sin\theta|}$$

alternatively use completenes of

$l=0$ $\quad Y_{00} = \frac{1}{\sqrt{4\pi}}$

$l=1$ $\quad \begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \end{cases}$

Summary

Thus solutions of Laplace's Equation in ~~cylindrical~~ spherical geometry are of the form (r, θ, φ)

$$\phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m} r^{\ell} + b_{\ell m} r^{-(\ell+1)} \right) P_\ell^m(\cos\theta)\, e^{im\varphi}$$

The combination $e^{im\phi} P_\ell^m(\cos\theta)$

is given a special name

"spherical harmonic"

$$Y_{\ell m}(\theta, \varphi) = \underbrace{\sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}}}\, e^{im\varphi} P_\ell^m(\cos\theta)$$

→ makes $Y_{\ell m}$ "orthonormal"

$$b_\ell = a^{\ell+1}(2\ell+1) V \int_0^1 dx\, P_\ell$$

$$\phi = \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \frac{P_\ell(\cos\theta)}{r^{\ell+1}}\, b_\ell$$

as $r \to \infty$ $\qquad$ $\ell = 1$ term dominates

$$\phi = \cancel{P_1} \frac{\overbrace{\cos\theta}^{P_1}}{r^2}\, a^2 V \frac{3}{2}$$

dipole moment

$$\phi = \frac{|p| \cos\theta}{r^2}$$

<u>Dipole field</u> (we will explain disa with

d [$\bullet +q$, $\bullet -q$ field of two point charges separated by ~~a~~ infinitesimally small distance d only term to survive $2q = l-1$ term

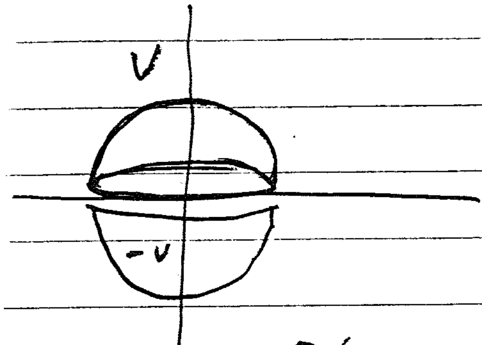
Thus l odd $\qquad q = \frac{l-1}{2}$

$$\int_0^1 dx P_l = -\frac{1}{2^l \cancel{l!}}\left[\frac{\cancel{l!}\,(l-1)!\,(-1)^{\frac{l+1}{2}}}{\left(\frac{l-1}{2}\right)!\,\underbrace{\left(l-\frac{l-1}{2}\right)!}_{\left(\frac{l+1}{2}\right)!}}\right]$$

$(l-1)!\,\cancel{l}\,2 \ldots\ldots$

$$\int_0^1 dx P_l = -\frac{1}{2^l}\;\frac{(l-1)!\,(-1)^{\frac{l+1}{2}}}{\left(\frac{l-1}{2}\right)!\left(\frac{l+1}{2}\right)!}$$

Example   azimuthal symmetry



$m=0$

$$\Phi = \sum_{l} \left(a_l r^l + b_l \frac{1}{r^{l+1}}\right) P_l(\cos\theta)$$

Have discarded $Q_l(\cos\theta)$ since blows up at $\theta=0,\pi$.

odd symmetry around $\theta=\frac{\pi}{2}$

$\rightarrow \cos\theta = x$ odd.

$\Rightarrow l$ odd.

Outside the sphere

$$a_l = 0$$

$$\Phi = \sum_{l} b_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

$$\Phi(a,\theta) = \sum_{l} b_l \frac{1}{a^{l+1}} P_l(\cos\theta)$$

normaliz of P_l

$$\int_{-1}^{1} d\cos\theta\, P_l(\cos\theta)\, \Phi(a,\theta) = \frac{b_l}{a^{l+1}} \frac{2}{2l+1}$$

$$V\left[\int_0^1 d\cos\theta\, P_l(\cos\theta) - \underbrace{\int_{-1}^{0} d\cos\theta\, P_l(\cos\theta)}_{0 \text{ for } l \text{ even}}\right] = \frac{2b_l}{a^{l+1}(2l+1)}$$

l odd

$$2V \underbrace{\int_0^1 dx\, P_l(x)}_{} = \frac{2b_l}{a^{l+1}(2l+1)}$$

$$\frac{1}{2^l l!} \frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l \Big|_0$$

$$\int_0^1 dx\, P_\ell(x) = \int_0^1 dz \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

$$= \frac{1}{2^\ell \ell!} \frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell \Big|_0^1 \leftarrow \text{value at } 1 = 0$$

$$= -\frac{1}{2^\ell \ell!} \frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell \Big|_{x=0}$$

Binomial Theorem $\quad (a+b)^\ell = \sum_{q=0}^{\ell} a^q b^{\ell-q} \frac{\ell!}{q!(\ell-q)!}$

$$\int_0^1 dx\, P_\ell(x) = -\frac{1}{2^\ell \ell!} \frac{d^{\ell-1}}{dx^{\ell-1}} \sum_{q=0}^{\ell} x^{2q}(-1)^{\ell-q} \frac{\ell!}{q!(\ell-q)!} \Bigg|_{x=0}$$

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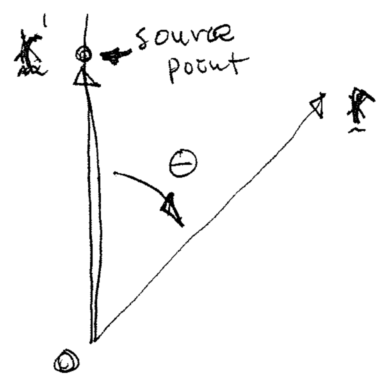
* Find Green's Function in Spherical Coordinates



$$\nabla^2 G(\underset{\sim}{r}, \underset{\sim}{r}') = -4\pi\,\delta(\underset{\sim}{r} - \underset{\sim}{r}')$$

observation point

We know the answer

$$G = \frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}$$

But also
$$G = \sum_\ell P_\ell(\cos\theta)\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right]$$

Can determine coefficients from dependence on r when $\theta = 0$

$$G = \frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} = \sum_\ell P_\ell(1)\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right]$$

$P_\ell(1) = 1$ by the way it is defind
see Rodriques formula

$$\frac{1}{|r - r'|} = \frac{1}{r}\left[\frac{1}{1 - r'/r}\right] = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^\ell \quad \text{if } r' < r$$

$$= \frac{1}{r'}\sum_{\ell=0}^{\infty}\left(\frac{r}{r'}\right)^\ell \quad \text{if } r < r'$$

Thus if $r' < r$ $\quad B_\ell = r'^{\,\ell} \quad A_\ell = 0$

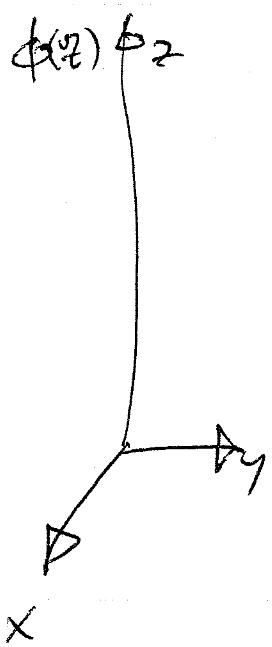
if $r' > r$ $\quad A_\ell = r'^{\,(\ell+1)} \quad B_\ell = 0$

$$\boxed{G = \sum P_\ell(\cos\theta)\,\frac{r_<^\ell}{\ldots}}$$ OK

In general, if $\phi(z)$ is known then in the form

$$\phi = \sum_{\ell=0}^{\infty}\left[A_\ell z^\ell + B_\ell z^{-(\ell+1)}\right]$$

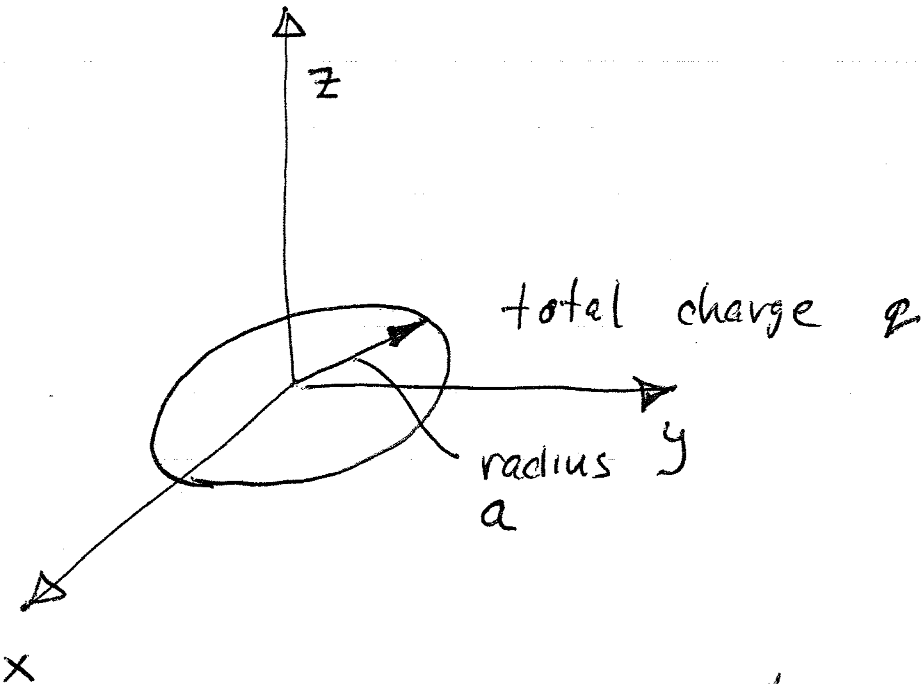
known



Then we can deduce

$$\phi(r,\theta) = \sum_\ell\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right]\frac{P_\ell(\cos\theta)}{P_\ell(1)}$$

Example, consider a ring charge q



$$\phi(z) = \frac{q/4\pi\epsilon_0}{\sqrt{z^2+a^2}}$$

Has a Taylor series expansion

where did this come from?

will show later

$$\frac{1}{\sqrt{z^2+a^2}} = \sum_{\substack{l=0 \\ l\ \text{even}}}^{\infty} P_l(0)\, \frac{z^l}{a^{l+1}} \qquad z<a$$

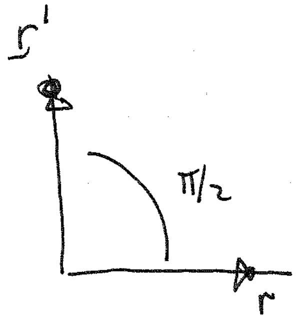
$$= \sum_{l=0}^{\infty} P_l(0)\, \frac{a^l}{z^{l+1}} \qquad z>0$$

THUS

$$\phi(r,\theta) = \frac{q}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(0) P_\ell(\cos\theta)$$

$r_>$ = greater of r and a $\qquad r_> = \begin{cases} r & \text{if } r > a \\ a & \text{if } a > r \end{cases}$

$r_<$ = lesser ~~smaller~~ of r and a



$$\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|}$$

$$= \sum \cdot P_\ell(0) \frac{r_<^{\ell}}{r_>^{\ell+1}}$$

$$= \frac{1}{\sqrt{r^2 + r'^2}}$$

Remember result for Green's function

$$\frac{1}{|\cancel{\mathbf{x}} - \cancel{\mathbf{x}}'|} = \sum_\ell P_\ell(\cos\gamma)\,\frac{r_<^{\ell}}{r_>^{\ell+1}}$$

$r_> =$

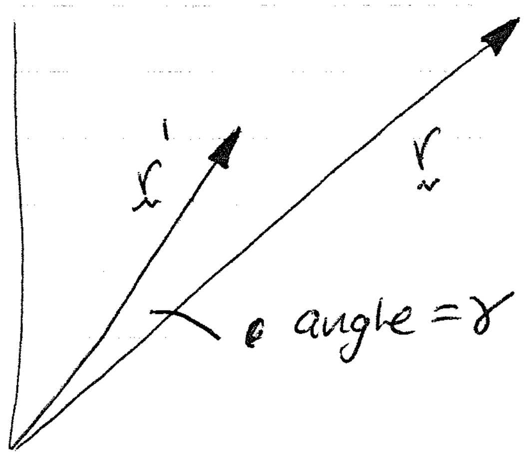
$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}$$

if $\cos\gamma = 0$

$$\boxed{\frac{1}{\sqrt{r^2+r'^2}} = \sum_\ell P_\ell(0)\,\frac{r_<^{\ell}}{r_>^{\ell+1}}}$$

can use this ~~to~~ as
~~define $P_\ell($~~
an alternate definition of P_ℓ

Addition theorem



$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi)$$

see book for proof

$$\boxed{\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')}$$

THUS Green's function for general spherical coordin

$$\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}}$$

$$= \sum_\ell P_\ell(\cos\gamma) \frac{r_<^\ell}{r_>^{\ell+1}}$$

~~= gre~~

$$= \sum_{\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{m=\ell} Y_{\ell m}^{*}(\theta', \varphi') Y_{\ell m}(\theta, \varphi)$$