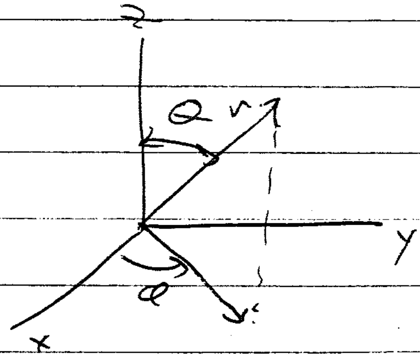
chapter 14

Basis Functions in Spherical Coordinates : $\nabla^2\varphi = 0$

Consider a basis function $\phi(r,\theta,\varphi)$



$$\nabla^2\phi = \frac{1}{r}\frac{\partial^2}{\partial r^2} r\phi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

$$\phi = \frac{U(r)}{r} P(\theta)\, \Phi(\varphi)$$

$$\frac{\partial^2\Phi}{\partial\varphi^2} + m^2\Phi = 0$$

Review this

$$\Phi = e^{\pm i m\varphi}$$

In a system where $\varphi \in (0, 2\pi)$, m is an integer.

$$\frac{r^2}{U}\frac{d^2U}{dr^2} + \boxed{\frac{1}{\sin\theta}\frac{1}{P}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} - m^2\frac{1}{\sin^2\theta}} = 0$$

$-l(l+1)$

$$\boxed{\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0}$$

$U = r^{l}, r^{l+1}$

$h = r^l, r^{-(l+1)}$

$$\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}P + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)P = 0\right.$$

near $\theta = 0$ soln $P_l \sim \theta^{\pm m}$

$\frac{1}{\theta}\frac{d}{d\theta}\theta\frac{d}{d\theta} - \frac{m^2}{\theta^2}P$

let $x = \cos\theta$

$dx = -\sin\theta\, d\theta$

Associated Legendre Functions

$$\boxed{\left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m = 0}$$

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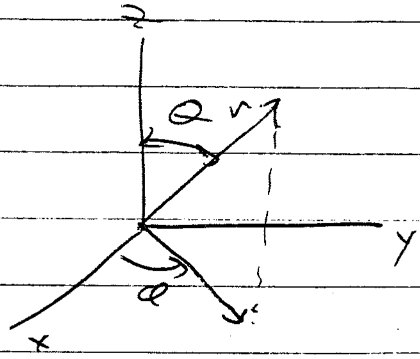

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial^2}{\partial r^2} r\phi + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

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$$\frac{r^2}{U}\frac{d^2U}{dr^2} + \boxed{\frac{1}{\sin\theta}\frac{1}{P}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} - m^2\frac{1}{\sin^2\theta}} = 0$$

$-\ell(\ell+1)$

$$\boxed{\frac{d^2U}{dr^2} - \frac{\ell(\ell+1)}{r^2}U = 0} \qquad U = r^{\ell},\ r^{-(\ell+1)}$$

$$h = r^{\ell},\ r^{-(\ell+1)}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}P + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right)P = 0$$

near $\theta = 0$ soln $P_\ell \sim \theta^{\pm m}$

let $x = \cos\theta$ $\qquad \frac{1}{\theta}\frac{d}{d\theta}\theta\frac{d}{d\theta} - \frac{m^2}{\theta^2}P$

$dx = -\sin\theta\, d\theta$

Associated Legendre Functions

$$\boxed{\left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + \ell(\ell+1) - \frac{m^2}{1-x^2}\right] P_\ell^m = 0}$$

Behaviour of Legendre Polynomials

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}P_\ell^m + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P_\ell^m = 0$$

OR $\quad x = \cos\theta$

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_\ell^m + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_\ell^m = 0$$

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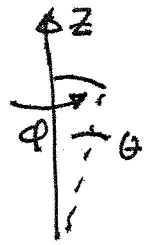
near $\theta = 0$ $\quad (x = 1)$ $\quad$ north pole



$\sin\theta \approx \theta$

neglect

$$\frac{1}{\theta}\frac{d}{d\theta}\theta\frac{d}{d\theta}P_\ell^m + \left[\ell(\ell+1) - \frac{m^2}{\theta^2}\right]P_\ell^m = 0$$

Possible behaviour

$\cancel{P_\ell^m \propto}$ $\quad \theta^{\pm m}$ $\quad$ keep $+m$

near $\theta = \pi/2$ $\quad$ (equator)

$\sin\theta \simeq 1$ $\quad \dfrac{d^2}{d\theta^2}P_\ell^m + (\ell(\ell+1) - m^2)P_\ell^m = 0$

ordinary point $\quad$ solution is analy.

It will be found that for

$$1 \geq |x| \geq 0 .$$

THAT for $P_\ell^m(x)$ to be well behaved (non singular)

ℓ must be a positive integer

m must be an integer satisfying

$$|m| \leq \ell .$$

This situation you probably have already encountered in Quantum Mechanics $\hbar \ell$ = total angular momentum

$\hbar m$ = component of angular momentum in 3 direction

$$\int_0^\pi \sin\theta\, d\theta \to \int_{-1}^1 dx$$

$$\int_{-1}^1 dx\, P_\ell^m \left[\frac{d}{dx}(1-x^2)\frac{d}{dx}P_\ell^m + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P_\ell^m\right]$$

$$= 0$$

$$\ell(\ell+1)\int_{-1}^1 dx\, |P_\ell^m|^2 = \int_{-1}^1 dx\left[(1-x^2)\left|\frac{dP_\ell^m}{dx}\right|^2 + \frac{m^2}{1-x^2}|P_\ell^m|^2\right]$$

Do by parts

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~~$\phi(x,y,z) = \sum_n x^n a_n + x^{n_1} y$~~

Taylor Expansion

$$\phi(x,y,z) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} x^{n-p} y^p a_{np}(z)$$

$$a_{00} + x a_{10} + y a_{11} + x^2 a_{20} + xy a_{21} + y^2 a_{22}$$

$$\cdots$$

~~$x = r\cos\theta \sin\phi$~~

put $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Thus term representing $e^{im\phi} \sim \begin{pmatrix} \sin\phi \\ \text{or} \\ \cos\phi \end{pmatrix}^m$

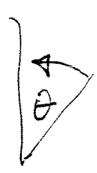
appears $n = m$

must have $r^m \sin^m\theta$

$$a_{np}(z) = \underline{a_{np}(0)} + \underline{a'_{np} z}$$

now expand around $z=0$

$\phi \sim \cancel{r^{m+1}}$ $\phi \sim r^l$



term with m must vary in r at least r^m

thus $l \geq m$

Series solution $(m=0)$

$$P_\ell(x) = \sum_{i=0}^{\infty} a_i x^{i}$$

solution is regular at $x = 0$

$x = \cos\theta$

insert in equation

$$\sum_{i=0}^{\infty} a_i x^{i-2} (i)(i-1)$$

$$- \sum_{i=0}^{\infty} a_i x^{i} \left[(i)(i+1) - \ell(\ell+1) \right]$$

equating equal powers gives

~~odd~~ must
odd are even odd

$i' = i+2$ $\qquad a_{i+2}\left((i+2)(i+1)\right) = a_i \left[i(i+1) - \ell(\ell+1) \right]$

1. 2, 3 4, 5 6 $\qquad$ 2-3 even ℓ

even and odd power decouple

even series $\quad a_0, a_2, a_4 \cdots$ $\qquad$ all odd $=0$ $\quad$ ~~$a_{odd} = 0$~~

odd series $\quad a_1, a_3, a_5 \cdots$ $\qquad$ all even $=0$

note as $i \to \infty$

$$\frac{a_{i+2}}{a_i} \to 1$$

Thus, series for P_ℓ will not converge ~~unless~~ for $x = 1$ unless ($\theta = $) it terminates at a specific i

This requires $\cancel{\boxed{\ell = \text{integer}}}$

$$\cancel{P_\ell(x) = \sum_{\substack{i=0 \\ i\ \text{even}}}^{\ell} a_i x^i}$$

suppose ℓ is even

even series terminates when $i = \ell = \text{even } \#$

odd series ~~$i = \ell + 1$~~ does not terminate

$$\cancel{i^2 + i = \ell(\ell+1)}$$

$$\cancel{i = -1 \pm \sqrt{1 + 4\ell(\ell+1)}}$$

$(1-z^2)^{m/2} \times$ Polynomial of order $\cancel{2}\ l-m$

Rodrigues Formula for $P_l^m(x)$

$(m>0)$

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

$$= \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$

for $|m| \leq l$
m negative as well

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

must be proportional

satisfy same eq & bc.

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l$$

suppose ℓ is odd vice verso

~~even~~ odd series terminate

even series not terminate

P_ℓ = series that terminates (regular at $x=1$)

normalization $P_\ell(1) = 1$

$P_0 = 1$

P

$P_1 = x$

$P_2 = \frac{1}{2}(3x^2 - 1)$

$\vdots$

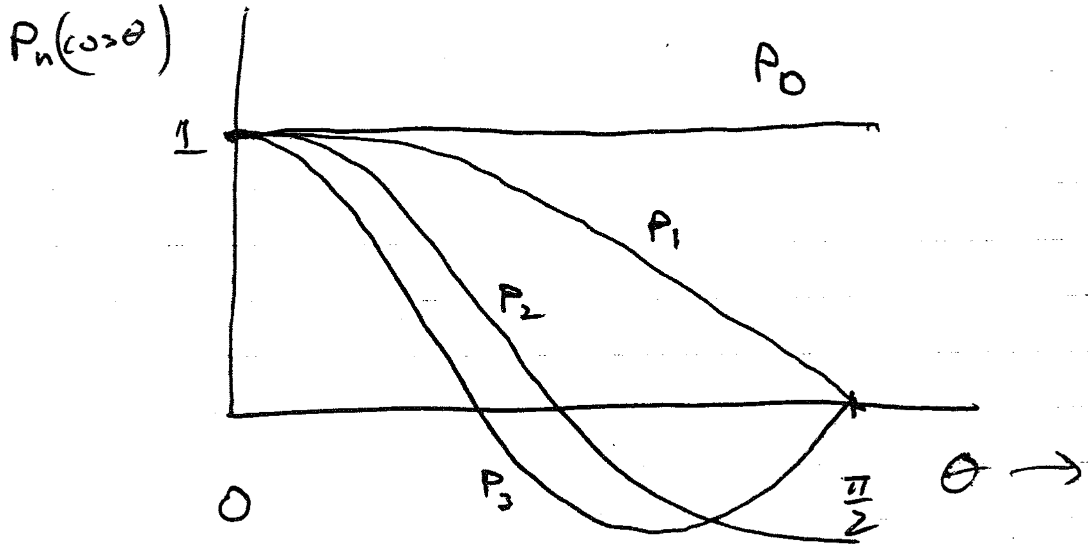
$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{(dx)^\ell} (x^2-1)^\ell$$

Rodriguez Formula

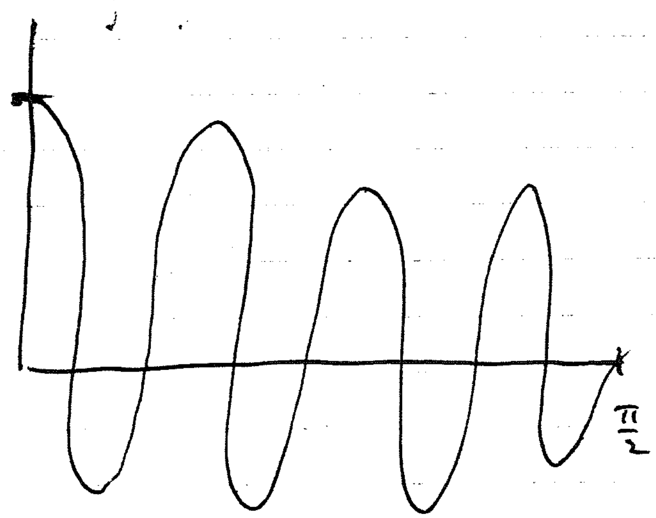
Q_ℓ = series that does not terminate

necessary if $x=1$ excluded.

$$P_0 = 1 \quad P_1 = \cos\theta$$
$$P_2 = \tfrac{1}{2}\left(3\cos^2\theta - 1\right)$$



High n



n large.

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_n + n(n+1)P_n = 0$$

WKB solution for $x \neq 1$

$$P_n \sim e^{\pm \int^x dx\, k(x)}$$

$$\boxed{k^2 = \frac{n(n+1)}{1-x^2}}$$ $\Rightarrow$ solution oscillates for n large

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orthogonality

$$\int_{-1}^{1} dx \quad P_{\ell'}^{m} \left[\frac{d}{dx}(1-x^2)\frac{d}{dx} P_{\ell}^{m} + \ell(\ell+1) - \frac{m^2}{1-x^2} \right] P_{\ell}^{m} = 0$$

do by parts different ℓ same m

$$\int_{-1}^{1} dx \quad P_{\ell}^{m} \left[\frac{d}{dx}(1-x^2)\frac{d}{dx} P_{\ell'}^{m} + \ell'(\ell'+1) - \frac{m^2}{1-x^2} \right] P_{\ell'}^{m} = 0$$

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subtract

$$\left[\ell(\ell+1) - \ell'(\ell'+1) \right] \int_{-1}^{1} dx \; P_{\ell'}^{m} P_{\ell}^{m}$$

$$+ \cancel{P_{\ell}^{m}} \; \underbrace{(1-x^2)}_{0} \underbrace{\left[P_{\ell'}^{m} \frac{dP_{\ell}^{m}}{dx} - P_{\ell}^{m} \frac{d}{dx} P_{\ell'}^{m} \right]}_{\text{regular at } x = \pm 1}\Bigg|_{-1}^{1} = 0$$

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thus $\int_{-1}^{1} dx\, P_{\ell'}^{m} P_{\ell}^{m} = 0$ unless $\ell' = \ell$

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