* Poisson Equation with Neumann boundary conditions

$$\nabla^2\phi = \frac{-\rho(\underline{x})}{\epsilon} \quad \text{in Volume} - V$$

$$n\cdot\nabla\phi\Big|_S \quad \text{specified on surface of } V$$

First: do solutions exist?

Ans: not necessarily.

~~From divergence~~
From Gauss's Law

$$\epsilon_0 \int_S da\, \underset{\sim}{n}\cdot\underline{E} = -\epsilon_0 \int_S da(n\cdot\nabla\phi) = q = \int_V d^3x\, \rho(\underline{x})$$

Thus, in order for a solution to exist one must specify values of $n\cdot\nabla\phi|_S$ that are consistent with Gauss's Law

Suppose this is the case. Then are solutions unique?

Ans: No they are not.

Suppose $\phi(x)$ satisfies equatons + BC.

Then $\phi(x) + \text{Const}$ also satisfies equation & BC's.

So to make solution unique we must add an additional condition.

Example $\displaystyle\int_S da\, \phi = 0$

The average value of ϕ on the surface is zero.

How to use Green's Theorem

$$-\nabla^2\phi = \frac{\rho(x)}{\epsilon_0} \qquad n\cdot\nabla\phi \text{ specified}$$

$$\nabla^2\psi = -4\pi\,\delta(\underline{x}-\underline{x}')$$

$$\phi(\underline{x}') = \int \frac{d^3x\,\rho(\underline{x})\,\psi(\underline{x},\underline{x}')}{4\pi\epsilon_0} - \frac{1}{4\pi}\int_S da\left[\phi\, n\cdot\nabla\psi - \psi\, n\cdot\nabla\phi\right]$$

Now to pick the BC's for ψ equation.

Let's pick $n\cdot\nabla\psi = K$ a constant

What is K?

$$\int_S da \underbrace{n\cdot\nabla\psi}_{K} = -4\pi\int_V d^3x\,\delta(\underline{x}-\underline{x}') = -4\pi$$

$$K = \frac{-4\pi}{A} \leftarrow \text{area of bounding surface}$$

$$\Phi(\underline{x}') = \int \frac{d^3x\, \rho(x)\, \psi}{4\pi\epsilon_0} - \frac{1}{4\pi}\int_S da\left[\overset{\text{const.}}{\phi K} - \psi\, \underbrace{n\cdot\nabla\phi}_{\text{known}}\right]$$

But $\int_S da\, \phi = 0$ to make solution unique

$$\phi(x') = \int \frac{d^3x\, \rho(x)\, \psi}{4\pi\epsilon_0} + \frac{1}{4\pi}\int da\, n\cdot\nabla\phi\, \psi(\underline{x}, x')$$