CHAPTER 8

Variational Solution

1. take a lucky guess

$$\phi = \phi_0 \sin\frac{\pi x}{d} \qquad \phi_0 \text{ unknown}$$

satisfies BC has one free parameter

$$I[\psi] = \int_0^d dx \left(\frac{1}{2}\frac{\pi^2}{d^2}\phi_0^2 \sin^2\frac{\pi x}{d} - \frac{\rho_0}{\epsilon_0}\phi_0 \sin^2\frac{\pi x}{d} \right)$$

$$= \frac{d}{2}\left[\frac{1}{2}\phi_0^2 \frac{\pi^2}{d^2} - \frac{\rho_0 \phi_0}{\epsilon_0} \right]$$

$$\int_0^x dx \sin^2\frac{\pi x}{d} = \frac{d}{2}$$

MINIMIZE $I(\psi)$ WRT ϕ_0 sel

$$\frac{dI}{d\phi_0} = \frac{d}{2}\left[\cancel{2}\phi_0 \frac{\pi^2}{d^2} - \frac{\rho_0}{\epsilon_0} \right] = 0$$

$$W = \int d^3x \frac{\epsilon_0 |\underset{\sim}{E}|^2}{\cancel{8\pi}\, 2}$$

$$w = \frac{\epsilon_0 |\underset{\sim}{E}|^2}{\cancel{8\pi}\, 2}$$

we can identify this as potential energy stored in the form of ~~electric~~ the electric field. One immediatey runs into trouble here if one reverts to point charges. Why ?

$$\int d^3x \frac{|\underset{\sim}{E}|^2}{8\pi} \to \infty$$

The electric field energy associated with a point charge is ∞

self energy is included

can compare difference of two states

for continuous charge density

$$W = \frac{1}{2}\int d^3x \int d^3x' \, \frac{\rho(x)\rho(x')}{4\pi\epsilon_0 |\underset{\sim}{x} - \underset{\sim}{x}'|}$$

(note no longer worry about $\underline{x} = \underline{x}'$

also can be written

$$W = \frac{1}{\cancel{8\pi}}\int d^3x \, \frac{\rho(\underset{\sim}{x})\phi(\underset{\sim}{x})}{2}$$

BUT $\qquad \rho(x) = -\frac{\epsilon_0}{\cancel{4\pi}}\nabla^2\phi(x)$

so

$$W = -\frac{1}{\cancel{8\pi}}\int d^3x \, \phi(x)\nabla^2\phi(x)\frac{\epsilon_0}{2}$$

$$= \int d^3x \, \frac{|\nabla\phi|^2}{\cancel{8\pi}}\frac{\epsilon_0}{2}$$

assuming $\phi \to 0$ as $x \to \infty$

Electrostatic potential Energy

consider $i = 1, j-1$ charges q_i at points $\underset{\sim}{x}_i$

now bring in the j th charge to point $\underset{\sim}{x}_j$ this takes an amount of work

$$W_j = q_j \phi(\underset{\sim}{x}_j)$$

$$W_j = q \sum_{i=1}^{j-1} \frac{q_j q_i}{|\underset{\sim}{x}_j - \underset{\sim}{x}_i|} \frac{1}{(4\pi\epsilon_0)}$$

THIS was the work required to bring the j th charge.

The work required ~~to~~ to assemble all ~~charges is~~ n is

$$W = \sum_{j=1}^{n} W_j = \sum_{j=1}^{n} \sum_{i=1}^{j-1} \frac{q_i q_j}{|\underset{\sim}{x}_j - \underset{\sim}{x}_i| \, 4\pi\epsilon_0}$$

$$= \frac{1}{2} \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{q_i q_j}{|\underset{\sim}{x}_j - \underset{\sim}{x}_i| \, 4\pi\epsilon_0},$$

double counting

$$= \sum_i \int_{S_i} ds \, \epsilon_0 \, \phi_0 \underbrace{\left(n \cdot \nabla \phi_{t,out} - n \cdot \nabla \phi_{t,in} \right)}$$

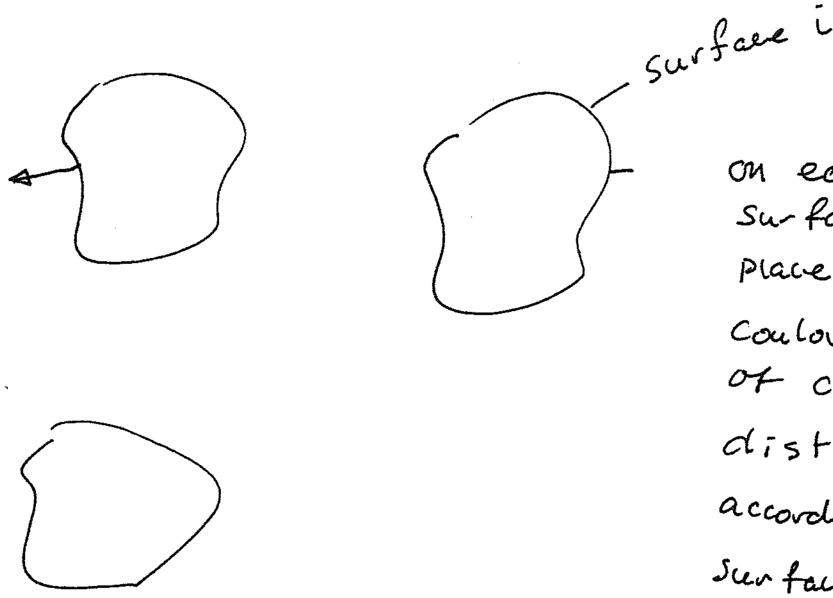
$$= \sum_i \int_{S_i} ds \; \phi_{0i} \, \sigma_{it} \qquad = 0$$

$$\int \sigma_{i0} \, ds = Q_i \qquad\qquad \int \sigma_{it} \, ds = 0$$

only way integral can be zero is if

$\phi_{0i} = \text{const}$

(1) **1.13** Thomson's



on each surface place Q_i Couloumbs of charge distributed according to surface charge density σ_i

Form of σ_i is unknown

$$\int_{S_i} dS\, \sigma_i = Q$$

Minimize

$$W = \int d^3x \frac{\epsilon_0 |\nabla\phi|^2}{2}$$

where $\nabla^2\phi = 0$ except at surfaces where

$$\sigma_i = \epsilon_0 \left[n\cdot\nabla(\phi_{out}) - n\cdot\nabla\phi_{in} \right]$$

let $\sigma_i = \sigma_{io} + \epsilon\, \sigma_{it}$ ← test

actual minimizing charge

2

$$W = \int_{V \text{ outside all}} d^3x\, \epsilon_0 \frac{|\nabla\phi|^2}{2} + \sum_i \int_{V_i} d^3x\, \epsilon_0 \frac{|\nabla\phi|^2}{2}$$

$$\cancel{W} = \int_V d^3x \frac{\epsilon_0}{2} \nabla\phi \cdot \nabla\phi$$

outward normal

$$= \int_S ds \frac{\epsilon_0}{2} \phi\, n \cdot \nabla\phi - \int_V d^3x \frac{\epsilon_0}{2} \phi \nabla^2\phi$$

For our problem $\nabla^2\phi = 0$

$$W = \sum_i \int_{S_i} ds \frac{\epsilon_0}{2} \phi \left(n \cdot \nabla\phi_{out} - n \cdot \nabla\phi_{in} \right)$$

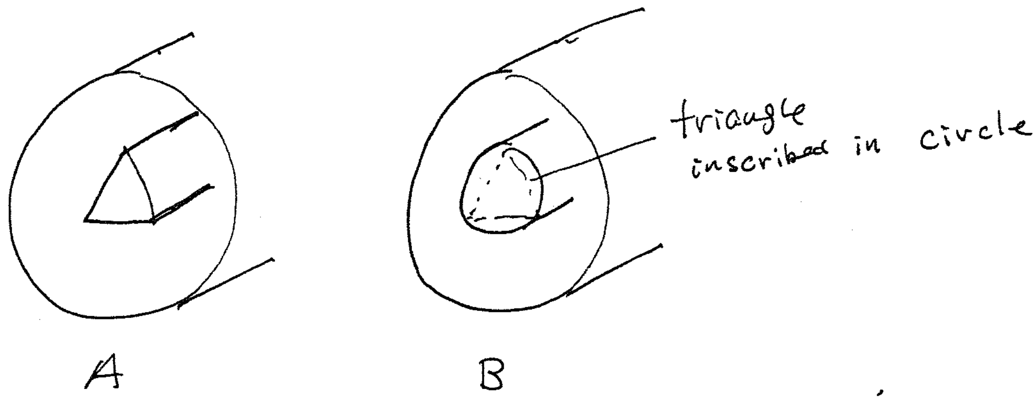
$$\phi = \phi_0 + \epsilon\phi_t$$

$$W(\epsilon) = \int_{\text{ALL-V}} d^3x \frac{\epsilon_0}{2} \left[|\nabla\phi_0|^2 + 2\epsilon \nabla\phi_0 \cdot \nabla\phi_t + \epsilon^2 |\nabla\phi_t|^2 \right]$$

$$\left.\frac{\partial W}{\partial \epsilon}\right|_{\epsilon=0} = \int_{\text{ALL} \cdot V} d^3x\, \epsilon_0 \nabla\phi_0 \cdot \nabla\phi_t$$

Examples of Applications

~~Thomson's Theorem~~



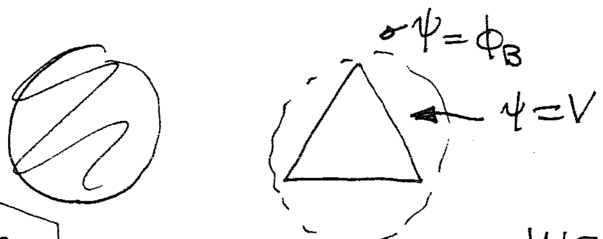
~~the~~ which one has a bigger Capacitance?

$$W = \frac{\epsilon_0}{2}\int d^3x\, |\nabla \phi|^2 \qquad \nabla^2\phi = 0 \qquad \phi = 0 \text{ on outer B}, \quad \phi = V \text{ inner B}$$

$$W = \frac{1}{2}CV^2$$

At fixed V bigger $W \rightarrow$ bigger C

For problem A let ψ be solution to problem B outside inner circle and $\psi = V$

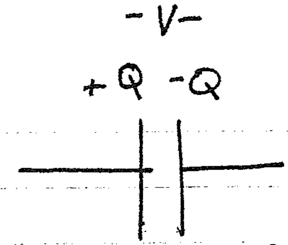


ψ is continuous
ψ is V on inner B
ψ is 0 on outer B

$$\boxed{C_B > C_A}$$

$$W = \frac{\epsilon_0}{2}\int d^3x\, |\nabla\psi|^2 > \frac{\epsilon_0}{2}\int d^3x\, |\nabla\phi_A|^2$$

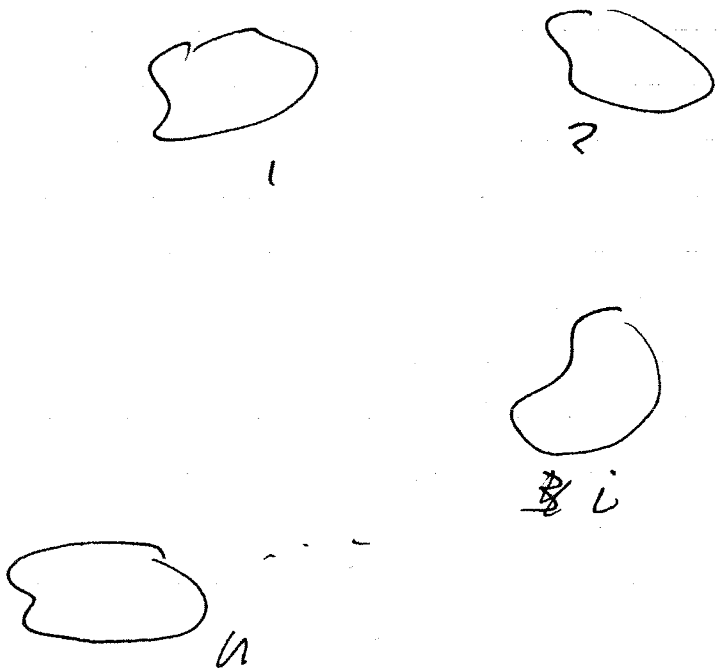
$$= \frac{V}{4}\epsilon_0\int d^3x\, |\nabla\phi_B|^2$$



$Q = VC$

~~Force on a~~

Capacitance: Consider a system of n conductors



on each conductor there is a surface charge density $\sigma_i(x)$ $i = 1, n$ and a total charge $Q_i = \int da\, \sigma_i(x)$

Assuming the conductors are

$\phi(\infty) = 0$

perfect conductors we have

$\phi(\underset{\sim}{x}) = V_i$ ~~at~~ a constant

on each conductor

THUS

$$W = \frac{1}{2}\int d^3x\, \rho(x)\phi(x) = \frac{1}{2}\sum_{i=1}^{n}\int da\, \sigma_i(x)\phi(x)$$

$$= \frac{1}{2}\sum_{i=1}^{n} V_i Q_i$$

$$\cancel{V_i = \phi(x) = \sum_{j=1}^{n}\int da\, \sigma_j(x)\,\frac{1}{|x|}}$$

~~$\underset{\sim}{x}$ at ith conductor~~

in general

inverse is C_{ij}

V_i is proportional to each of the Q_j & vice verse

~~capacitance matrix~~

$$V_i = \sum_j M_{ij} Q_j$$

Reciprocity $M_{ij} = M_{ji}$

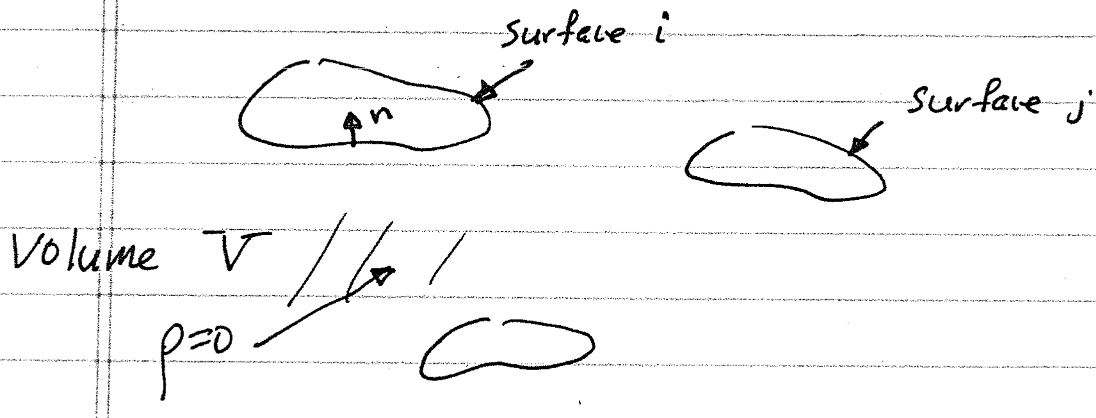
$$Q_i = \sum_j C_{ij} V_j$$

C_{ij} capacitance matrix

Follows from Green's theorem

$$\phi(\underline{x}') = \int \frac{d^3x}{4\pi\epsilon_0} \overset{\to 0}{\rho(\underline{x})}\, G_D(\underline{x}, \underline{x}') - \frac{1}{4\pi} \int_S dA\, \phi(\underline{x})\, \underline{n} \cdot \nabla G_D(\underline{x}, \underline{x}')$$

outward normal



$$Q_i = \int_{S_i} \epsilon_0 \underline{E}(\underline{x}') \cdot (-\underline{n})\, dA' = \int_{S'} \epsilon_0\, \underline{n} \cdot \nabla' \phi(\underline{x}')\, dA'$$

$$\underline{n} \cdot \nabla' \phi(\underline{x}') = -\frac{1}{4\pi} \sum_j V_j \int_{S_j} dA\, \underline{n} \cdot \nabla \left(\underline{n} \cdot \nabla' G_D(\underline{x}, \underline{x}') \right)$$

$$Q_i = \sum_j V_j \underbrace{\left(-\frac{1}{4\pi} \int_{S_i} dA' \int_{S_j} dA \;\; \underset{\sim}{n} \cdot \nabla \left(\underline{n} \cdot \nabla G_D(\underline{x}, \underline{x}') \right) \right)}_{C_{ij}}$$

$C_{ij} = C_{ji}$ because $G_D(\underline{x}, x') = G_D(\underline{x}', x)$

OPTIONAL

CONSIDER GREEN's THEOREM FOR PROBLEM WITH conductors

$$\phi(\underset{\sim}{x}) = \int d^3x' \, G_D(\underset{\sim}{x}, \underset{\sim}{x}')\, \rho(\underset{\sim}{x}')/4\pi\epsilon_0$$

$$- \frac{1}{4\pi} \sum_i \int_{S_i} da' \, \phi(\underset{\sim}{x}')\, \underset{\sim}{n}\cdot\nabla' G_D(\underset{\sim}{x}, \underset{\sim}{x}')$$

on S_i $\quad \phi(\underset{\sim}{x}') = V_i$

$\underset{\sim}{n}$ points out of V

$$Q_j = \epsilon_0 \cancel{\tfrac{1}{4\pi}} \int_{S_j} da \, \underset{\sim}{n}\cdot\nabla\phi(\underset{\sim}{x})$$

$$Q_j = -\frac{\epsilon_0}{(4\pi)} \sum_i \overbrace{\int_{S_i} da' \int_{S_j} da \, \underset{\sim}{n}\cdot\nabla\, \underset{\sim}{n}\cdot\nabla' G_D(\underset{\sim}{x}, \underset{\sim}{x}')}^{C_{ji}} \, V_i$$

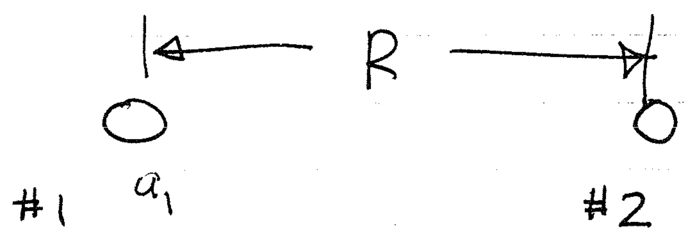
symmetric

$$Q_j = \sum_i M_{ji} V_i$$

capacitance matrix

implies $\quad V_i = \sum_j C_{ij} Q_j$

Example, calculate the <u>capacitance matrix</u> ~~due to~~ for two small spheres of radius a_1, a_2 ~~and~~ separated by a distance R with $R \gg a_1, a_2$



$R \gg a$ allows us to neglect the rearrangement of charge on spheres due to presence of other spheres.

Calculate potential due to point charge

$$V \cancel{\Phi}_1 = \frac{Q_1}{4\pi\epsilon_0 a_1} + \frac{Q_2}{4\pi\epsilon_0 R} \qquad V\cancel{\Phi}_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{R} + \frac{Q_2}{a_2}\right)$$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{1}{4\pi\epsilon_0}\begin{bmatrix} 1/a_1 & 1/R \\ 1/R & 1/a_2 \end{bmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

Invert

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \frac{4\pi\epsilon_0}{\frac{1}{a_1 a_2} - \frac{1}{R^2}}\begin{bmatrix} \frac{1}{a_2} & -\frac{1}{R} \\ -\frac{1}{R} & \frac{1}{a_1} \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

neglect

C_{11} $\quad$ C_{12}

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = 4\pi\epsilon_0\begin{bmatrix} a_1 & -a_1 a_2/R \\ -a_1 a_2/R & a_2 \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

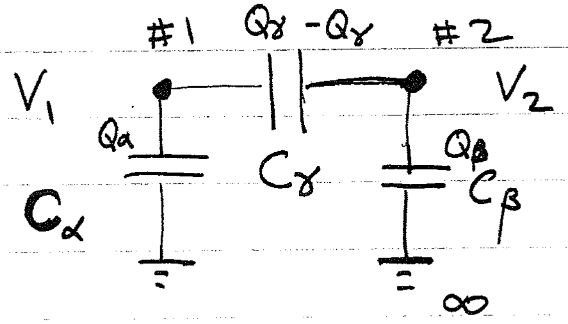
C_{22}

Cap

Symmetric

SKIP

~~what is the capacitance~~



Equivalent circuit

$$Q_\gamma = C_\gamma (V_1 - V_2)$$

$$Q_\alpha = C_\alpha V_1$$

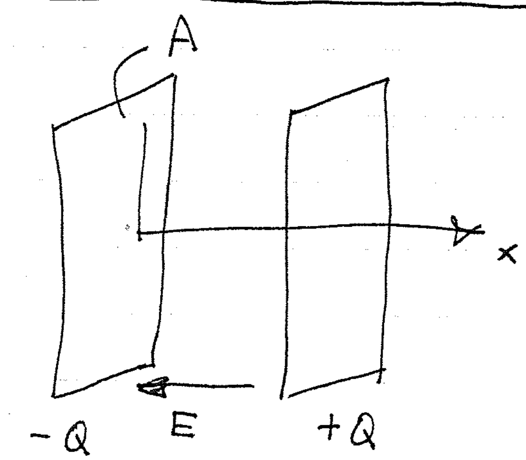
$$Q_\beta = C_\beta V_2$$

$$Q_1 = (Q_\alpha + Q_\gamma) = C_\alpha V_1 + C_\gamma (V_1 - V_2)$$

$$= \overset{C_{11}}{(C_\alpha + C_\gamma)} V_1 - \overset{C_{12}}{C_\gamma} V_2$$

$$Q_2 = (Q_\beta + -Q_\gamma) = (\cancel{C_\beta +} \overset{C_{21}}{-C_\gamma} V_1 + \overset{C_{22}}{(C_\beta + C_\gamma)} V_2$$

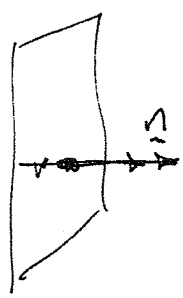
FORCE on a parallel plates



surface charge densit

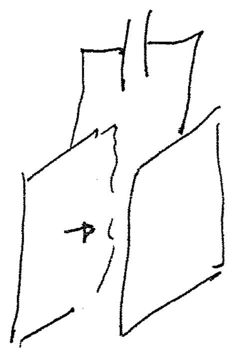
$$\underset{\sim}{E} = -4\pi \frac{Q}{A}\, \underset{\sim}{a}_x$$

Force on plate $\quad \underset{\sim}{F} = \int da\, \underset{\sim}{n}\, \frac{|E_n|^2}{8\pi}$



attraction to other plate

Check out Problem 1.9 use virtual work



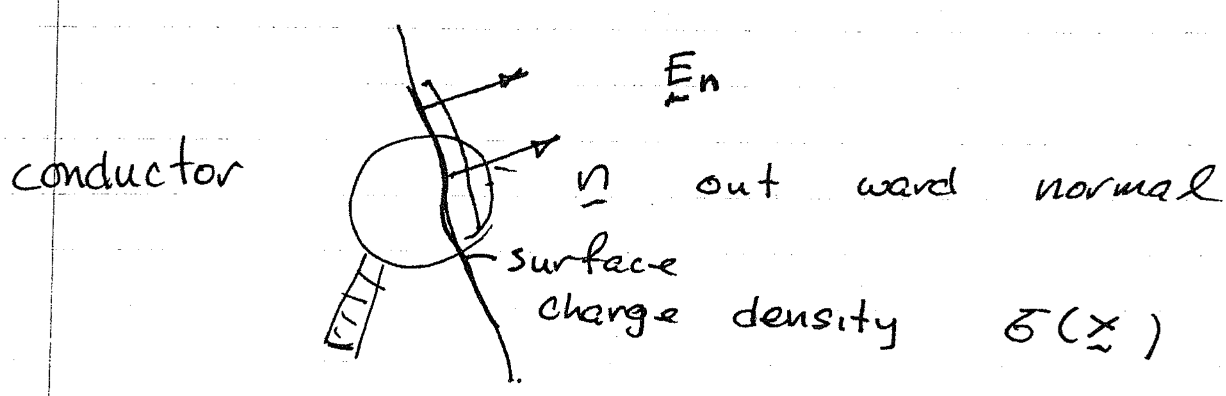
constan C

$$\delta W = \underset{\sim}{F}\cdot\delta \underset{\sim}{X} = \delta\left[\tfrac{1}{2}CV^2 \text{ or } \frac{Q^2}{2C}\right]$$

constant V

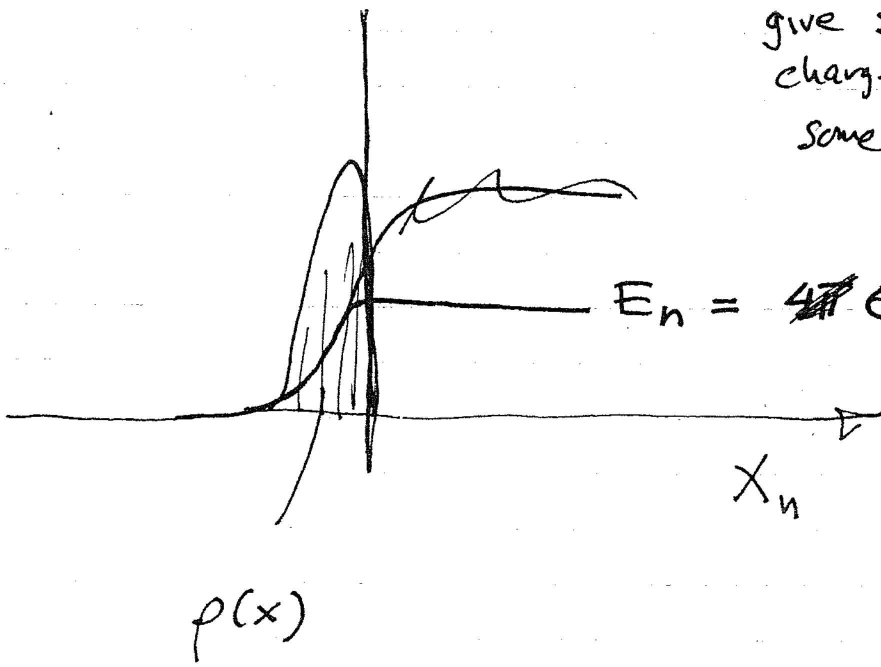
must include $V\delta Q$

Force on the surface of a conductor



Blow up of surface

give surface charge densit some ~~th~~ thickner



$$E_n = \cancel{4\pi}\,\sigma = \cancel{4\pi}\int dx_n \frac{\rho(\underline{x})}{\epsilon_0}$$

$$\frac{\partial E_n}{\partial x_n} = \cancel{4\pi}\frac{\rho(x_n)}{\epsilon_0} \qquad \sigma = \int dx_n\, \rho(\underline{x})$$

small force act

$$\cancel{d}F_x =$$

$$\underset{\sim}{F} = \int d^3x \, \rho(\underset{\sim}{x}) \, \underset{\sim}{E}$$

normal direction

$$= \int da^{\underset{\sim}{n}} \int dx_n \, \rho \, E_n$$

$$\frac{\partial E_n}{\partial x_n} = \cancel{4\pi}\rho/\epsilon_0$$

$$= \int da \, \underset{\sim}{n} \int \frac{\epsilon_0}{\cancel{4\pi}} E_n \frac{\partial E_n(x_n)}{\partial x_n}$$

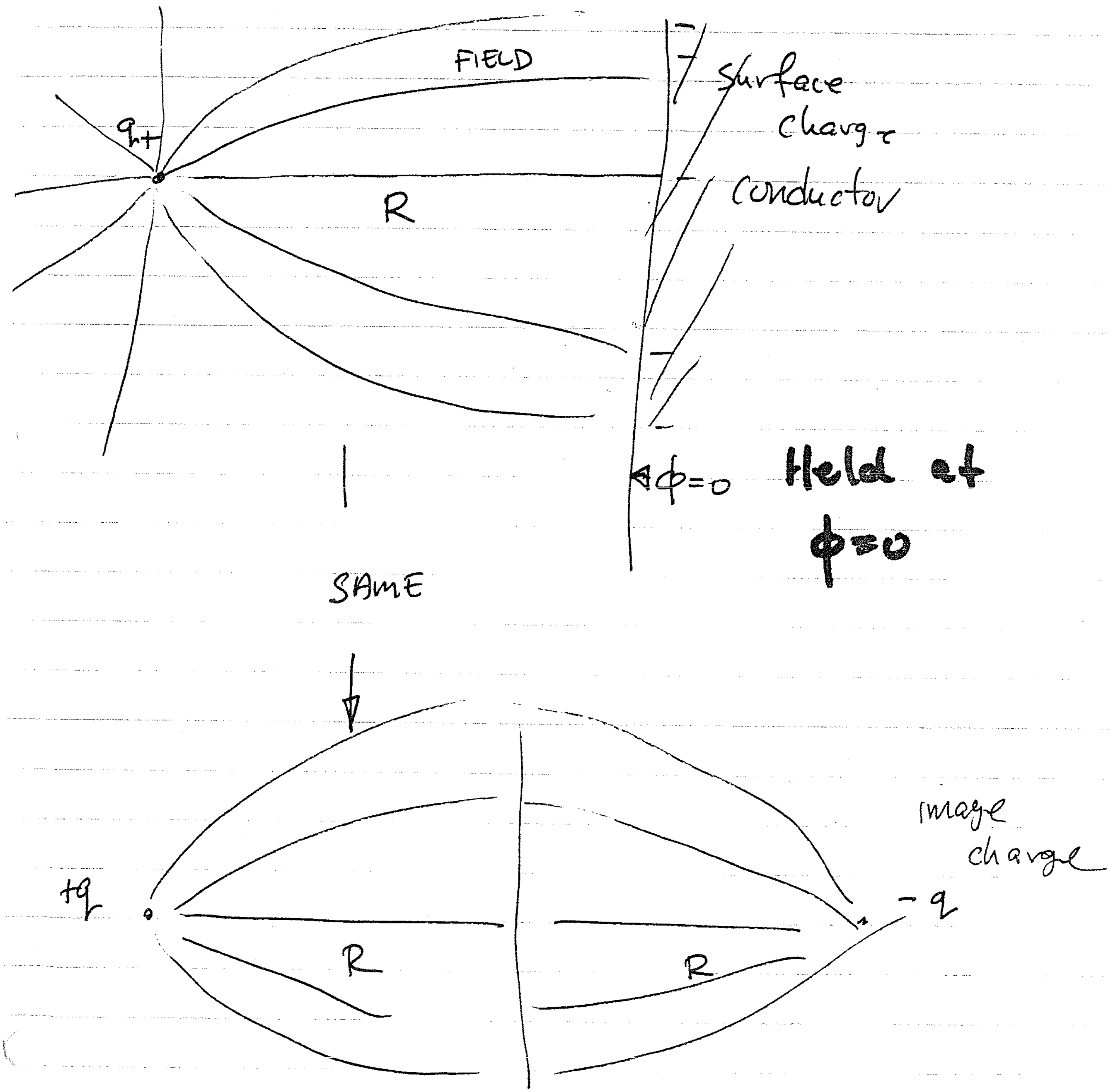
$$= \int da \, \underset{\sim}{n} \, \frac{\epsilon_0}{\cancel{8\pi}} \frac{|E_n|_v^2}{2}$$

out.

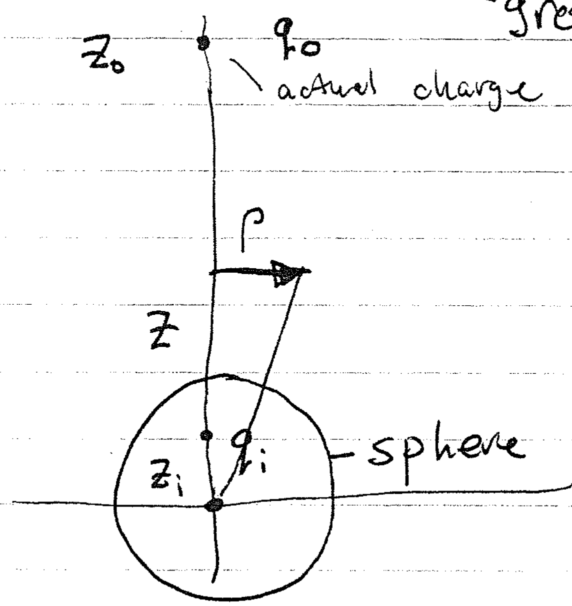
electric pressure $= -\frac{\epsilon_0}{\cancel{8\pi}} \frac{|\underset{\sim}{E}|_v^2}{2}$

CHAPTER 9

II Method of images works with sufficient symmetry



suppose we wish to find image charge for a conducting sphere grounded ($\phi = 0$)



can I find a q_i & z_i such that $\phi = 0$ on surface of sphere of radius a

Use polar coordinates

potential due to act / im

$$\phi(\rho, z) = \frac{q_0}{\sqrt{\rho^2 + (z - z_0)^2}} + \frac{q_i}{\sqrt{\rho^2 + (z - z_i)^2}}$$

$= 0$ on sphere $\quad q_0 q_i < 0$ opposite

FIND LOCUS OF POINTS FOR $\phi = 0$

$$q_0^2 \left(\rho^2 + z^2 + z_i^2 - 2 z z_i \right) - q_i^2 \left(\rho^2 + z^2 + z_0^2 - 2 z z_0 \right) = 0$$

We want this to describe a sphere of radius a

vs $\boxed{\rho^2 + z^2 - a^2 = 0}$

THIS REQUIRES

$$q_0^2 z_i - q_i^2 z_0 = 0 \qquad \text{TERMS LINEAR IN } z \text{ VANISH}$$

$$q_0^2 z_i^2 - q_i^2 z_0^2 = -(q_0^2 - q_i^2) a^2$$

$$\cancel{z_i = \frac{q_i^2 z_0}{q_0^2}}$$

$$\cancel{q_0^2 \left(\frac{q_i^2}{q_0^2}\right)^2 z_0^2 - \frac{q_i^2}{q_0^2} z_0^2 = \left(1 - \frac{q_i^2}{q_0^2}\right) a^2}$$

$$\boxed{\begin{array}{l} \text{Solution} \quad z_i z_0 = a^2 \quad \text{geometric mean} \\ q_i = -q_0\, a / z_0 \end{array}}$$

note as $z_0 \to a$

$z_i \to a$ $\quad q_i \to -q_0$

let $z_0 = a + \epsilon$

$\epsilon \ll a$

then $z_i \simeq a - \epsilon$

$q_i \simeq -q_0$ images close to plane

calculate surface charge density

$$\phi = \frac{q_0}{|\underset{\sim}{r} - \underset{\sim}{e}_z z_0|} + \frac{q_i}{|\underset{\sim}{r} - \underset{\sim}{e}_z z_i|}$$

normal electric field

$$E_r = - \underset{\sim}{e}_r \cdot \nabla\phi \Big|_{|\underset{\sim}{r}|=a} = \sigma/\epsilon_0$$

$$= + \underset{\sim}{e}_r \cdot \left[\frac{q_0 (\underset{\sim}{r} - \underset{\sim}{e}_z z_0)}{|\underset{\sim}{r} - \underset{\sim}{e}_z z_0|^3} + \frac{q_i (\underset{\sim}{r} - \underset{\sim}{e}_z z_i)}{|\underset{\sim}{r} - \underset{\sim}{e}_z z_i|^3} \right]$$

note, on sphere $\underset{\sim}{r} = a\underset{\sim}{e}_r$

$$\frac{q_0}{|\underset{\sim}{r} - e_z z_0|} + \frac{q_i}{|\underset{\sim}{r} - e_z z_i|} = 0$$

$$\frac{1}{|\underset{\sim}{r} - \underset{\sim}{e}_z z_i|^3} = -\frac{q_0^3}{q_i^3 |\underset{\sim}{r} - e_z z_0|^3}$$

$$|\underset{\sim}{r} - e_z z_i| = - |\underset{\sim}{r} - e_z z_0| \frac{q_0}{q_i}$$

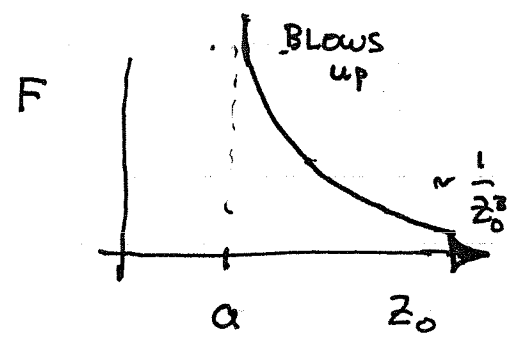
Calculate force ~~on~~ of attraction between charge ~~due~~ and ~~to~~ sphere. FORCE on q_0 same as produced by image charge

force on q_0

$$\underset{\sim}{F} = \frac{q_0 q_i \hat{e}_z (z_0 - z_i)}{|z_0 - z_i|^3}$$

where $z_i = a^2/z_0$

$q_i = -q_0 \, a/z_0$



$$\underset{\sim}{F} = -e_z \left(\frac{q_0^2}{z_0^2}\right) \frac{a}{z_0(1 - a^2/z_0^2)^2}$$

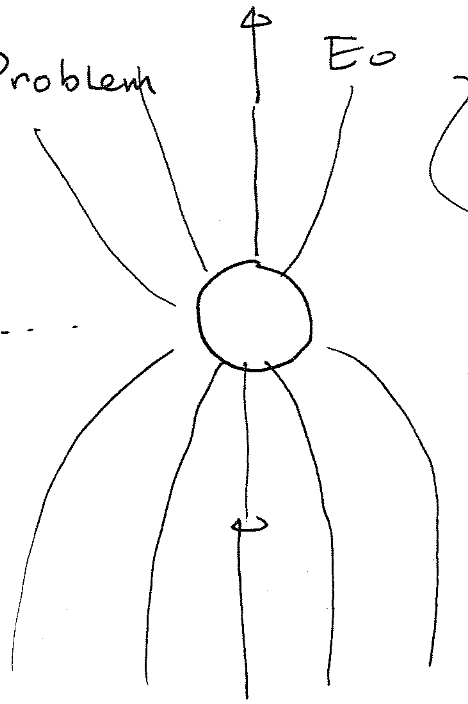
can also show that it is

the same as

$$F_{sphere} = \frac{1}{2}\int da' \sigma \, \frac{q_0(\underset{\sim}{r} - z_0 e_z)}{|\underset{\sim}{r} - z_0 \underset{\sim}{e}_z|^3} = -$$

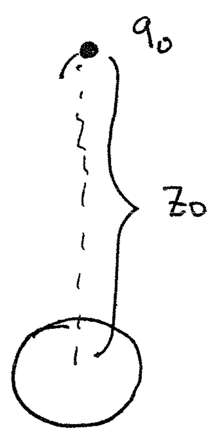
1

Problem



uniform electric field $\underset{\sim}{E}_0$

metal sphere of radius a



without sphere

$$\underset{\sim}{E}(0) = -\frac{q_0}{4\pi\epsilon_0 z_0^2}\hat{z}$$

let $q_0 = -4\pi\epsilon_0 E_0 z_0^2$

and $z_0 \to \infty$

dipole moment of sphere

$$\underset{\sim}{p} = \hat{z}\, q_i z_i = \hat{z}\left(-\frac{q_0 a}{z_0}\right)\frac{a^2}{z_0}$$

induced dipole moment

$$\boxed{\underset{\sim}{p} = -a^3 \hat{z}\, \frac{q_0}{z_0^2} = 4\pi\epsilon_0 E_0 a^3 \hat{z}}$$

Remember the role of $G_D(\underset{\sim}{x}, \underset{\sim}{x}')$

specified charge density outside sphere

$$\phi(\underset{\sim}{x}) = \int \frac{d^3x' \, G_D(\underset{\sim}{x}, \underset{\sim}{x}') \rho(\underset{\sim}{x}')}{4\pi\epsilon_0}$$

$\underset{\sim}{n}$ is normal pointing out of V (into sphere)

$$-\frac{1}{4\pi} \int_S da' \, \underset{\sim}{n} \cdot \nabla' G_D(\underset{\sim}{x}, \underset{\sim}{x}') \phi(\underset{\sim}{x}')$$

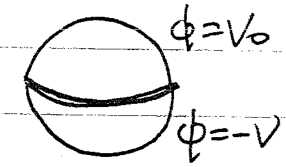
potential on sphere

We are now in a position to solve a host of problems

Eample: $\rho(x) = 0$ ϕ specified on boundary of sphere

$$\phi(\underset{\sim}{x}) = -\frac{1}{4\pi} \int_S da' \, \phi(\underset{\sim}{x}') \, \underset{\sim}{n} \cdot \nabla' G_D(\underset{\sim}{x}, \underset{\sim}{x}')$$

<u>See book</u>



CHAPTER 10

$q_0 - q_i$

$$E_r = \frac{+\,\underset{\sim}{e}_r \cdot}{4\pi\epsilon_0 |\underset{\sim}{r} - \underset{\sim}{e}_z z_0|^3} \left[q_0 (\underset{\sim}{r} - e_z z_0) - q_i (\underset{\sim}{r} - e_z z_i) \frac{q_0^3}{q_i^3} \right]$$

$|\underset{\sim}{r}| = a$

$$= + \cancel{\frac{q_0}{4\pi\epsilon_0 |\underset{\sim}{r} - e_z z_0|^3} \; a \ldots z_0 - (1 - \frac{q_0^2}{q_i^2})}$$

substitute $q_i^2 = q_0^2 a^2 / z_0^2$

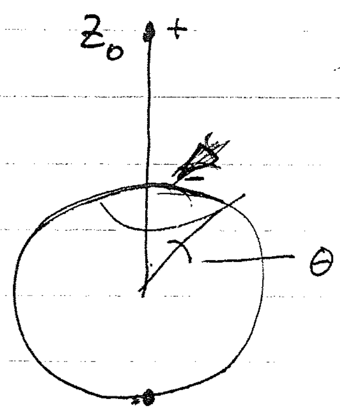
$$= + \frac{q_0}{4\pi\epsilon_0 |\underset{\sim}{r} - \underset{\sim}{e}_z z_0|^3} \left\{ a\left(1 - \frac{q_0^2}{q_i^2}\right) - \underset{\sim}{e}_r \cdot e_z \underbrace{\left(q_0 z_0 - \frac{q_0^3}{q_i^2} z_i \right)}_{0} \right\}$$

$\underset{\sim}{r} = a\,\underset{\sim}{e}_r$

$$|\underset{\sim}{r} - \underset{\sim}{e}_z z_0| = \sqrt{a^2 + z_0^2 - 2 z_0 a \cos\theta}$$

$$\cancel{4\pi}\,\sigma = \frac{E_r}{4\pi}$$

σ largest when



$q_0 z_0 - \frac{q_0^3}{q_i^2}$

$$z_i = \frac{a^2}{z_0}$$

~~Ex~~

smallest

$$\cancel{\sigma = -\frac{q_0 a(1}{4\pi}}$$

$$E_r = -\frac{q_0}{|r}$$

$$E_r = \frac{q_0}{4\pi\epsilon_0 |\underset{\sim}{r} - \underset{\sim}{e}_z Z_0|^3}$$

$$\left\{ a\left(1 - \frac{q_0^2}{q_i^2}\right) - \underset{\sim}{e}_r \cdot \underset{\sim}{e}_z \left(q_0 Z_0 - \frac{q_0^3}{q_i^3} Z_i \right) \right\}$$

SUBSTITUTE $\quad q_i = -q_0 \left(\frac{a}{Z_0}\right)$

$$Z_i = \frac{a^2}{Z_0}$$

$$1 - \frac{q_0^2}{q_i^2} = 1 - \frac{Z_0^2}{a^2}$$

$$q_0 Z_0 - \frac{q_0^3}{q_i^2} Z_i = q_0 Z_0 - \frac{q_0^3 a^2 Z_0^2}{Z_0 q_0^2 a^2} = 0$$

$$E_r = \frac{q_0}{4\pi\epsilon_0 |\underset{\sim}{r} - \underset{\sim}{e}_z Z_0|^3} a\left(1 - \frac{Z_0^2}{a^2}\right)$$

$$|\underset{\sim}{r} - e_z Z_0| = \sqrt{r^2 + Z_0^2 - 2 r Z_0 \cos\theta}$$

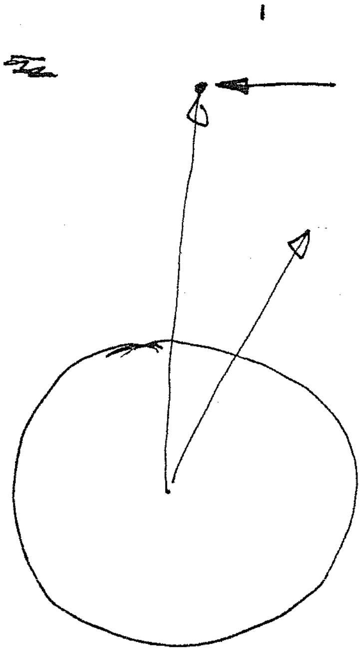
$$\nabla^2 G_D = -4\pi\,\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$

$\phi = 0$ on sphere

DIRICHILET Green's FUNCTION = POTENTIAL due to ~~point cha~~ unit charge

$q_0 = 1$

$G_D(\underset{\sim}{x}, \underset{\sim}{x}') =$



sorce at $\underset{\sim}{x}'$

observation point at $\underset{\sim}{x}$

4π

$q_0 = 1 \qquad q_i = -\dfrac{a}{|\underset{\sim}{x}'|}\ 4\pi$

$\underset{\sim}{r}_0 = \underset{\sim}{x}' \qquad \underset{\sim}{r}_i = \underset{\sim}{x}' \dfrac{a^2}{|\underset{\sim}{x}'|^2}$

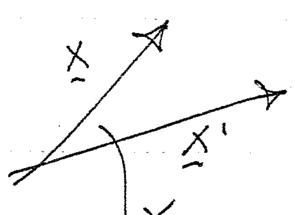
$\underset{\sim}{r} = \underset{\sim}{x}$

$$G_D(\underset{\sim}{x}, \underset{\sim}{x}') = \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} - \frac{a}{|\underset{\sim}{x}'|}\,\frac{1}{\left|\underset{\sim}{x} - \underset{\sim}{x}'\dfrac{a^2}{|\underset{\sim}{x}'|^2}\right|}$$

G_D

Symmetric? yes

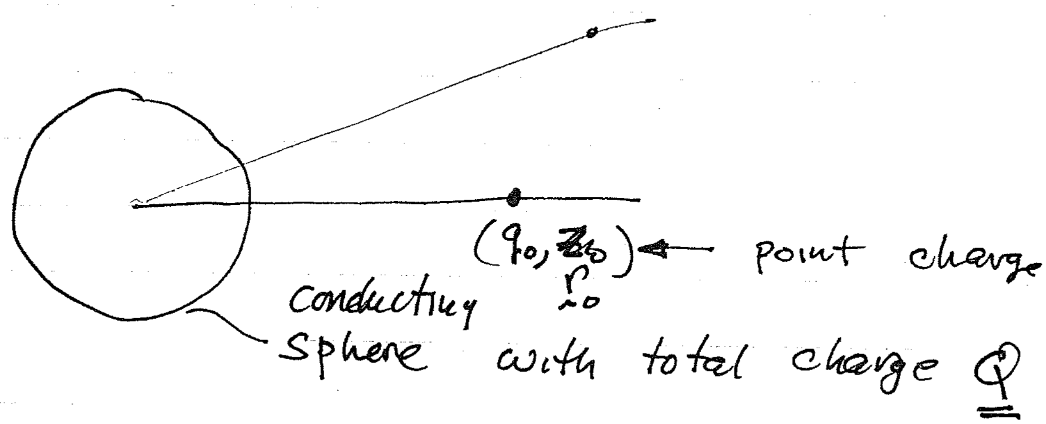
$$\left|\underset{\sim}{x} - \underset{\sim}{x}'\frac{a^2}{|x'|^2}\right| = \sqrt{x^2 + x'^2\left(\frac{a^2}{x'^2}\right)^2 - 2xx'\frac{a^2}{x'^2}\cos\gamma}$$



$$= \frac{a}{x'}\sqrt{\frac{x^2x'^2}{a^2} + a^2 - 2xx'\cos\gamma}$$

$$G_D = \frac{1}{\ldots} - \frac{1}{\ldots}$$

Method of images used to find field surrounding a arbitrarily charged ~~spere~~ sphere



Let $\phi_0(\underset{\sim}{r})$ be the solution we have obtained previously corresponding to a grounded sphere

note generalization

$$\phi_0(\underset{\sim}{r}) = \frac{q_0}{4\pi\epsilon_d|\underset{\sim}{r}-\underset{\sim}{r}_0|} + \frac{q_i}{4\pi\epsilon_d|\underset{\sim}{r}-\underset{\sim}{r}_i|}$$

$$\boxed{q_i = -q_0\frac{a}{|\underset{\sim}{r}_0|} \qquad \underset{\sim}{r}_i = \underset{\sim}{r}_0\frac{a^2}{|\underset{\sim}{r}_0|^2}}$$

This satisfies

$$\phi_0(|\underset{\sim}{r}|=a) = 0$$

the total charge on the sphere is q_i.

To obtain a solution for an arbitrary amount of charge add to this the solution corresponding to a uniform charge distribution, on surface (same as a point charge

$$\phi(\underset{\sim}{r}) = \phi_0(\underset{\sim}{r}) + \frac{(Q-q_i)}{4\pi\epsilon_0|\underset{\sim}{r}|}$$

The sum corresponds to a total charge Q

satisfies Poisson equation outside sphere

$$\nabla^2\phi = -4\pi\,\delta(\underset{\sim}{r}-\underset{\sim}{r}_0)\,q_0$$

$\phi =$ const on surface of sphere.

unique solution