CHAPTER 4

Lecture #2 507

Boundary Conditions on ~~&~~ surfaces

~~consider~~

$$\phi(x) = \int \frac{d^3x' \, \rho(\underset{\sim}{x}')}{4\pi\epsilon_0 |\underset{\sim}{x} - \underset{\sim}{x}'|}$$

1.6
1.7
1.8
1.9
1.10
1.11

$$\underset{\sim}{E} = -\nabla\phi(\underset{\sim}{x}) \Rightarrow \oint \underset{\sim}{E} \cdot d\underline{l} = 0$$

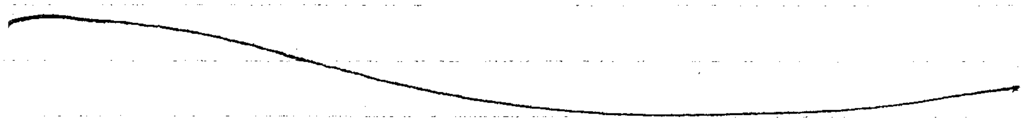
represents a complete solution ~~for~~ for the electric field given the charge density. Problem is we do not always know the charge density before hand, but we must sometimes determine it as part of the solution of the problem.

~~To do this we then require~~

~~ru~~

For example suppose we introduce an perfect electrical conductor into our problem. Charge on such a conductor is free to move and arrange it self such that $\underset{\sim}{E} = 0$ within the conductor. Once $\underset{\sim}{E} = 0$ there no longe will be any electrical force on a free charge <u>within</u> the conductor inducing the charge to move. (This does not imply that there is no force on the conductor as we shall see) vacuum

$\underset{\sim}{E} \neq 0$



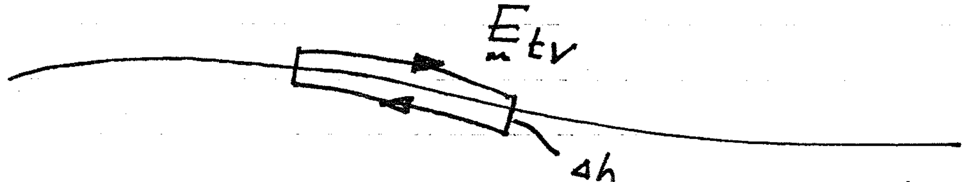
conductor

$\underset{\sim}{E} = \underset{\sim}{0}$

In conductor $\quad \nabla \cdot \underset{\sim}{E} = \nabla \cdot \underset{\sim}{0} = 0$

no net charge density inside conductor

At boundary



conductor

$\oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = 0$

closed loop

tangential component

This implies $\underset{\sim}{E}_{tv} - \underset{\sim}{E}_{tc} = 0$

but $\underset{\sim}{E}_{tc} = 0$ ($\underset{\sim}{E} = 0$ in cond.

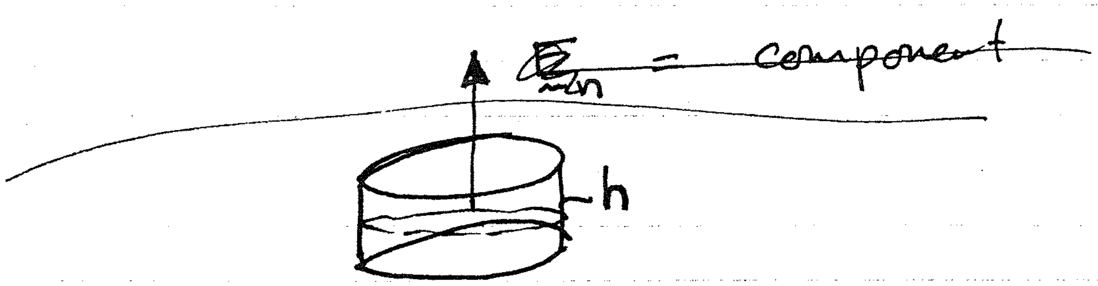
thus $\underset{\sim}{E}_{tr} = 0$ (two components)

$$\phi(1) - \phi(2) = -\int_1^2 \underset{\sim}{E}\cdot d\underset{\sim}{l}$$

taking path to lie in the surface implies that $\boxed{\phi = \text{const}}$ on surface of conductor.

~~this will be used to determine surface d~~

Normal Component



$$\epsilon_0 \int da\, \underset{\sim}{n}\cdot\underset{\sim}{E} = \cancel{4\pi}\, q_{enclosed}$$

$\underset{\sim}{E} = 0$ inside

outward

surface charge density

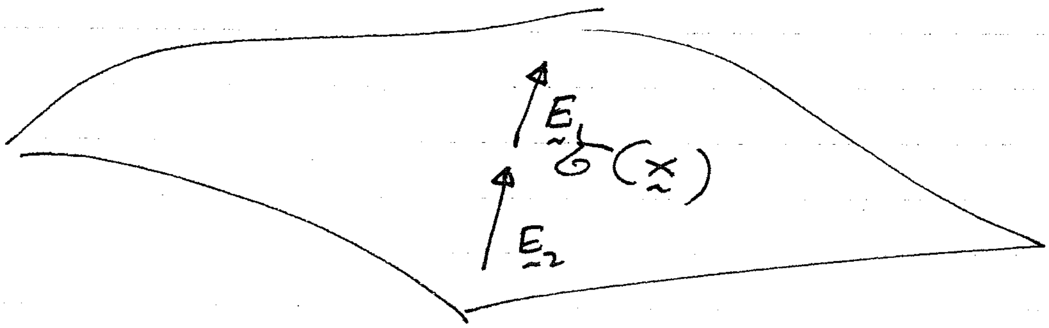
$h \to 0$

$da \to 0$

$$\underset{\sim}{n}\cdot\underset{\sim}{E} = \cancel{4\pi}\left(\frac{q_{enclosed}}{\epsilon_0\, da}\right)$$

$\sigma =$ surface charge density

~~THUS,~~



in general there is a discontinuity in normal component of E field at a surface with surface charge density

& $(E_1 - E_2)_{\text{tangent}} = 0$

normal pointing from $2 \to 1$

$$(E_1 - E_2)\cdot n = \cancel{4\pi}\,\sigma/\epsilon_0$$

& for surface charge density at a conductor $E_2 = 0$

Skip

Potential due to surface

Charge density

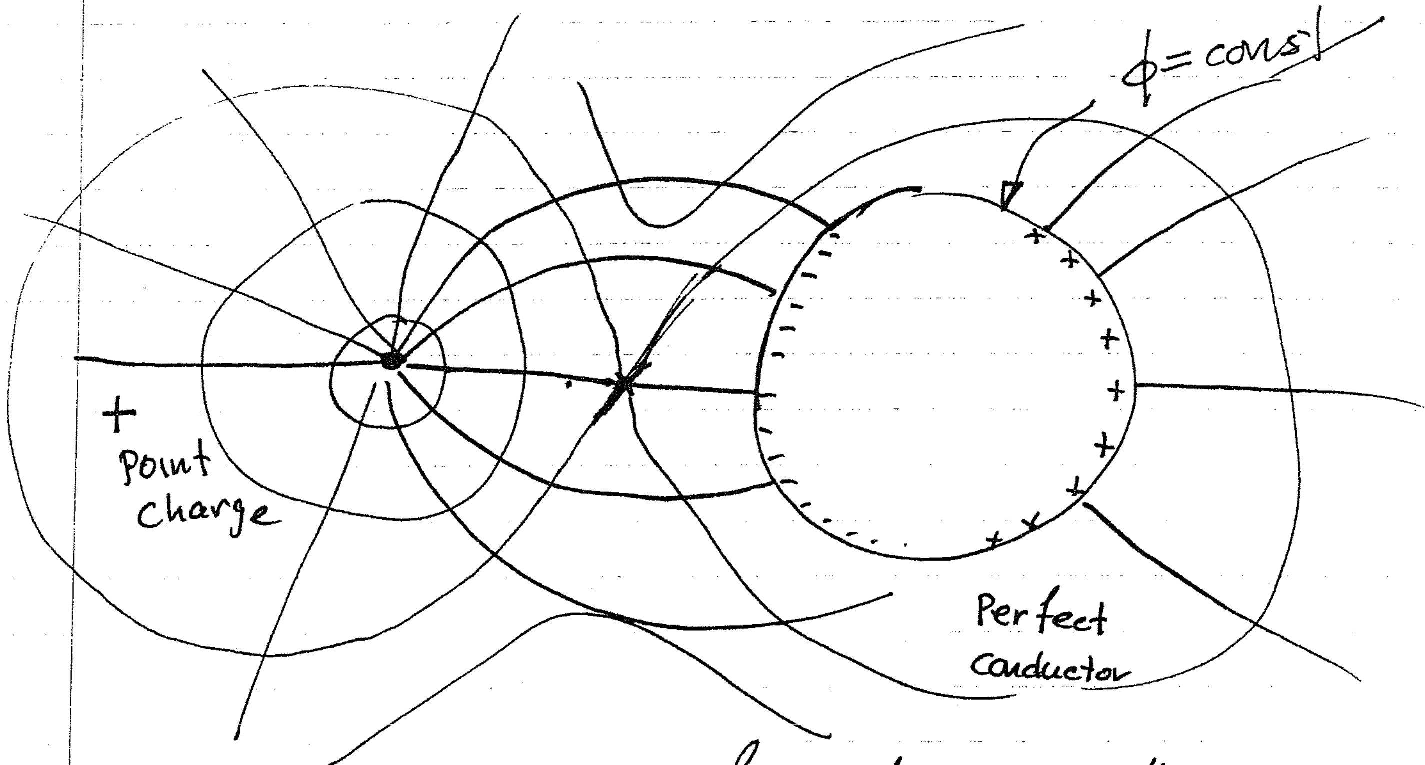
$$\phi(x) = \int_S \frac{\sigma(x') \, da'}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

~~Dipo~~ continuous

~~Other special charge distributions~~

OTHER DISTRIBUTIONS dipole we will come to this,

Caution while above is true you almost never know $\sigma(\underset{\sim}{x}')$ apriori suppose we move a ~~ex~~ perfect conductor into the picture



free charge within conductor arranges it self so that $\underset{\sim}{E} = 0$ inside conductor

Thus what we want to solve is

$$\epsilon_0 \nabla^2 \phi = -\cancel{4\pi}\rho(\underset{\sim}{x})$$

everywhere but in conductor in vacuum

~~with ϕ = const on conductor~~

~~total charge on~~

~~CONSIDER~~

~~with eil~~

Thus, the kind of problem we want to solve is

$$\epsilon_0 \nabla^2 \phi = - \cancel{4\pi} \rho(\underset{\sim}{x})$$

specified charge

does not include surface charge on boundary

along with certain boundary conditions on ϕ or normal derivative on given surfaces

ϕ specified on surface

"DIRICHILET" (usual case if conductors) present

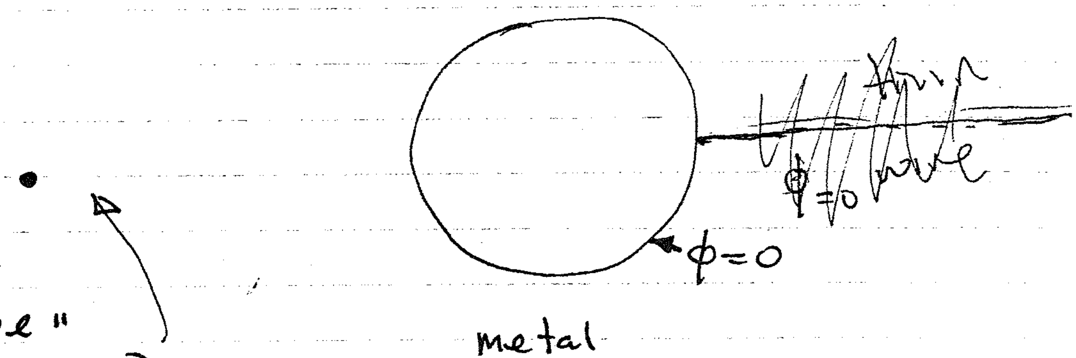
$\underset{\sim}{n} \cdot \nabla\phi = \frac{\partial \phi}{\partial n}$ specified "Neumann"

if surface charge

Caution

FOR poisson can not specify both on all surfaces.

switch to uniqueness



"point charge" or $\rho(\underset{\sim}{x})$

$\phi = 0$

metal conductor moved in ~~slowly~~ from ∞

What boundary conditions ?

$\phi =$ cons   potential held ~~at~~ fixed

$$\int_S da\, \underset{\sim}{n} \cdot \nabla \phi = \text{specified}$$

~~(no net~~ surface charge)

fixed charge on sphere

~~Green's Functions~~ Delta-functions

CONSIDER POISSON'S EQUATION

$$\epsilon_0 \nabla^2 \phi = - \cancel{4\pi}\rho(\underset{\sim}{x})$$

which has the solution

$$\phi(\underset{\sim}{x}) = \int d^3x' \frac{\rho(\underset{\sim}{x}')}{4\pi\epsilon_0 |\underset{\sim}{x} - \underset{\sim}{x}'|}$$

THIS MEANS

$$\nabla^2 \phi(x) = \int d^3x' \frac{\rho(x')}{4\pi\epsilon_0} \nabla^2 \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$$= - \frac{4\pi \rho(\underset{\sim}{x})}{\epsilon_0}$$

or $$\rho(\underset{\sim}{x}) = \int d^3x' \rho(\underset{\sim}{x}') \underbrace{\left[-\frac{1}{4\pi}\nabla^2 \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\right]}$$

True for any $\rho(\underset{\sim}{x})$

A VERY SPECIAL FUNCTION

$$\delta(\underset{\sim}{x}-\underset{\sim}{x}') = -\frac{1}{4\pi}\nabla^2 \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

$\stackrel{\text{def}}{=}$!

$$\cancel{\delta(\underset{\sim}{x}-\underset{\sim}{x}') = \begin{cases} 0 & \text{if } \underset{\sim}{x} \neq \underset{\sim}{x}' \\ \infty & \text{if } \underset{\sim}{x} = \underset{\sim}{x}' \end{cases}}$$

DIRAC DELTA FUNCTION

$\cancel{\int d^3x}$

$$\delta(\underset{\sim}{x}) = \begin{cases} 0 & \text{if } \underset{\sim}{x} \neq 0 \\ \infty & \text{if } \underset{\sim}{x} = 0 \end{cases}$$

$\cancel{\delta(\underset{\sim}{z})}$ −

but $$\int d^3x\, \delta(\underset{\sim}{x}) = \begin{cases} \underline{1} \end{cases}$$

TWO USEFUL RELATIONS

$$\nabla_x \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = \frac{-(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}$$

$$\nabla \cdot \nabla \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = \nabla \cdot \left(- \frac{(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \right) = -4\pi \overset{\text{vector}}{\delta}(\underset{\sim}{x}-\underset{\sim}{x}')$$

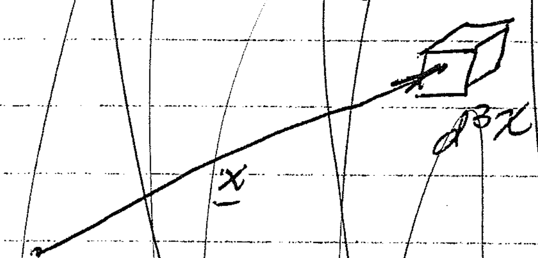
SKIP

Exp. fact is that forces superpose.

Superposition

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{(\underline{x}-\underline{x}_i)}{|\underline{x}-\underline{x}_i|^3}$$ for n point charges

Volume charge density



$$dq = \rho(\underline{x})\, d^3x$$

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\underline{x}') \frac{(\underline{x}-\underline{x}')}{|\underline{x}-\underline{x}'|^3}\, d^3x'$$

DIRAC DELTA FUNCTIONS

Allows us to express pt. charges as volume charge density

$$\rho(\underline{x}) = \sum_{i=1}^{n} q_i\, \delta(\underline{x}-\underline{x}_i)$$

where

$$\int_{\Delta V} \delta(\underline{x}-\underline{X})\, d^3x \equiv \begin{cases} 1 & \text{if } \underline{X} \text{ in } \Delta V \\ 0 & \text{if } \underline{X} \text{ not in } \Delta V \end{cases}$$

Properties of 1D delta functions

units of δ depends on the argument

(i) $\delta(x-a) = 0$ if $x \neq a$.

(ii) $\int_b^c \delta(x-a)\,dx = \begin{cases} 1 & \text{if } b<a<c \\ 0 & \text{if } a<b \text{ or } a>c \end{cases}$

undefined otherwise.

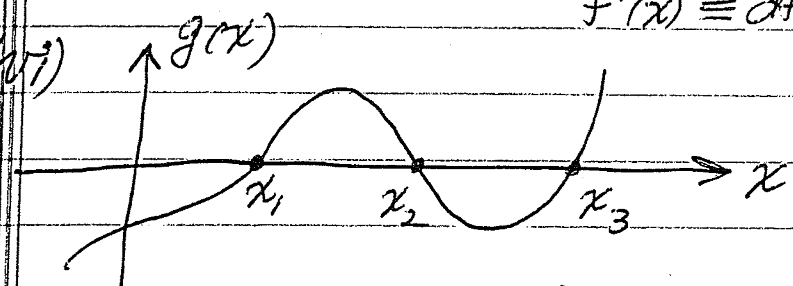
(iii) $\int_{-\infty}^{+\infty} f(x)\,\delta(x-a) = f(a)$ (assumes $f(x)$ is continuous at $x=a$).

(iv) $\int_{-\infty}^{+\infty} f(x)\,\delta'(x-a)\,dx = -f'(a)$ (by parts)

(assumes $f(x)$ is differentiable at $x=a$)

$f'(x) \equiv df/dx$

(vi)



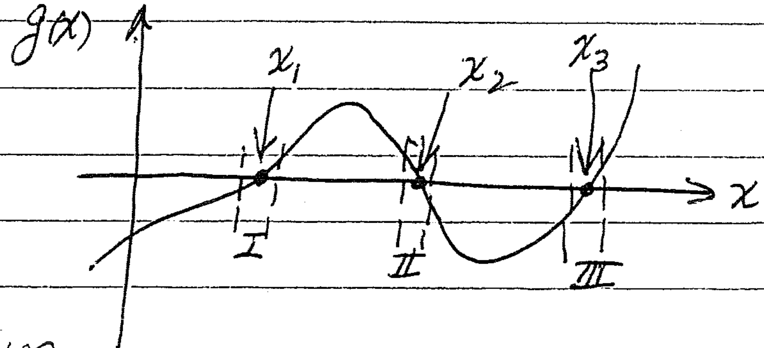
$g(x)$ has 3 zeros $(n=3)$

$$\delta(g(x)) = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|g'(x_i)|}$$

(v) $\delta(cx) = \frac{1}{|c|}\,\delta(x)$. (from (ii) $\int_{-\infty}^{+\infty} \delta(cx)\,dx = \frac{1}{|c|}$)

~~use change of~~ Variables $y = cx$

<u>Derivation</u> of (vi)

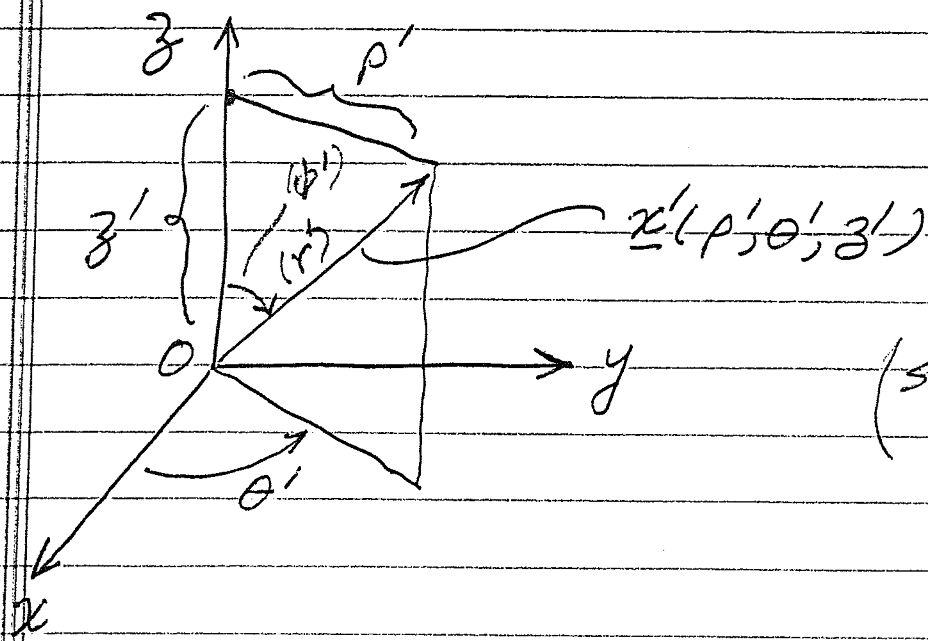


$$\int_{-\infty}^{+\infty} = \int_I + \int_{II} + \int_{III}$$

$\int_I \delta(g(x))\,dx = \int_{x_1-\epsilon}^{x_1+\epsilon} \delta[(x-x_1)g'(x_1)]\,dx = \frac{1}{|g'(x_1)|}$

5

Example: Represent the three dimensional delta function $\delta(\underline{x}-\underline{x}')$ in cylindrical coordinates.



(spherical: r', θ', ϕ')

↑ add to fig. when spherical mentioned.

$$\delta(\underline{x}-\underline{x}') = \frac{\delta(\rho-\rho')}{\rho'}\delta(\theta-\theta')\delta(z-z')$$

$$d^3x' = \rho' d\rho' d\theta' dz'$$

So $\int_V \delta(\underline{x}-\underline{x}')d^3x' = 1$ if (ρ', θ', z') is in V

In general

$$d^3x = \underbrace{h_1 h_2 h_3}_{\text{metric coeffs.}} \underbrace{du_1 du_2 du_3}$$ — (u_1, u_2, u_3): orthogonal coordinate system

$$\delta(\underline{x}-\underline{x}') = \frac{\delta(u_1-u_1')\delta(u_2-u_2')\delta(u_3-u_3')}{h_1 h_2 h_3}$$

E.g. in cylindrical coordinates.

$u_1 = \rho$, $u_2 = \theta$, $u_3 = z$ $\quad h_3 = r\sin\theta$

$h_1 = 1$, $h_2 = \rho$, $h_3 = 1$

Spherical coordinates $\quad dV = r^2 dr\, d\theta \sin\theta\, d\theta\, d\phi$

$u_1 = r$ $\quad u_2 = \theta$ $\quad u_3 = \phi$

Green's Theorem and Green's Function

consider Poisson's equation

$$\epsilon_0 \nabla^2 \phi = - \cancel{4\pi} \rho(\underset{\sim}{x})$$

and it's "solution"

$$\phi(\underset{\sim}{x}) = \int d^3x' \frac{\rho(\underset{\sim}{x}')}{4\pi\epsilon_0 |\underset{\sim}{x} - \underset{\sim}{x}'|}$$

~~THIS IS A FORM o~~

THIS IS an example of ~~one can think of the~~ a Green's function solution.

TRUE FOR any choice of ϕ & ψ

NOW LET ϕ be the solution to the problem we are interested.

$$-\nabla^2\phi = \frac{\rho(\underset{\sim}{x})}{\epsilon}$$ (haven't specified BC's

LET $\psi(\underset{\sim}{x})$ be the solution to

$$\nabla^2\psi = -4\pi\,\delta(\underset{\sim}{x}-\underset{\sim}{x}')$$ (haven't specified BC's)

(example $\psi = \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$)

$$\int_V d^3x \left[\phi(x)(-4\pi\,\delta(x-\underset{\sim}{x}')) + \psi(\underset{\sim}{x},\underset{\sim}{x}')\frac{\rho(\underset{\sim}{x})}{\epsilon_0} \right]$$

$$= \int_S da \left[\phi\, n\cdot\nabla\psi - \psi\, n\cdot\nabla\phi \right]$$

FIRST TERM $= -4\pi\,\phi(\underset{\sim}{x}')$ if x' is in V

$= 0$ otherwise

assume $\underline{x}'$ in V

$$\phi(\underline{x}') = \int \frac{d^3x \, \rho(x)}{4\pi\epsilon_0} \psi(\underline{x}, \underline{x}')$$

$$- \frac{1}{4\pi} \int_S da \left[\phi \, n \cdot \nabla \psi - \psi \, n \cdot \nabla \phi \right]$$

~~Boundary condit~~

Boundary Conditions

(dirichlet problem)

suppose $\phi(\underline{x})$ is known on boundary
(then $n \cdot \nabla \phi$ is unknown)

pick $\psi = G(\underline{x}, \underline{x}')$
so that $G\Big|_{X = \text{Boundary}} = 0$
call G_D

$$\phi(x') = \underbrace{\int \frac{d^3x \, \rho(\underline{x})}{4\pi\epsilon_0} G_D(\underline{x}, \underline{x}')}$$

particular solution satisfying $\phi = 0$ on Boundary

$$\underbrace{- \frac{1}{4\pi} \int_S da \, \phi(\underline{x}) \, n \cdot \nabla G_D(\underline{x}, \underline{x}')}$$

Homogeneous term from potential on boundary symmetry of

$$+\nabla^2 G_D = -4\pi\,\delta(\underline{x}-\underline{x}')$$

$$G_D(\underline{x},\underline{x}')\Big|_{\underline{x}=\text{Boundary}} = 0$$

$$-\nabla^2\phi = \frac{\rho(\underline{x})}{\epsilon_0} \qquad \phi=0 \quad \text{on } B$$

$$\phi(\underline{x}') = \frac{1}{4\pi\epsilon_0}\int_V d^3x\,\rho(\underline{x})\,G_D(\underline{x},\underline{x}')$$

NOW suppose $\rho(\underline{x}) = 4\pi\epsilon_0\,\delta(\underline{x}-\underline{x}'')$

THEN $\phi(\underline{x}) = G_D(\underline{x},\underline{x}'')$

$$\phi(\underline{x}') = G_D(\underline{x}'',x')$$

eliminate prime

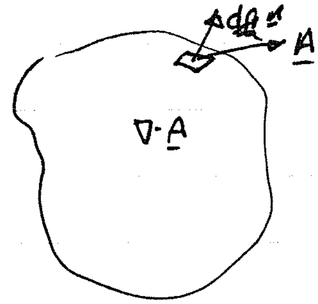
$$\phi(\underline{x}) = G_D(\underline{x}'',x)$$

arguments reversed (reciprocity)

Green's Theorem

Follows from divergence theorem

$$\int_V d^3x \, \nabla \cdot \underset{\sim}{A} = \int_S \underset{\sim}{A} \cdot n \, da$$



Let $\underset{\sim}{A} = \phi \nabla \psi$

ϕ & ψ are two arbitrary scalars

$$\nabla \cdot \underset{\sim}{A} = \nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi$$ product rule

$$\int_V \left[\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \right] d^3x = \int_S \phi \, n \cdot \nabla \psi \, da$$

interchange $\phi \leftrightarrow \psi$

$$\int_V \left(\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi \right) d^3x = \int_S \psi \, n \cdot \nabla \phi \, da$$

subtract

$$\boxed{\int_V \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi \right] d^3x = \int_S \left[\phi \, n \cdot \nabla \psi - \psi \, n \cdot \nabla \phi \right] da}$$

CHAPTER 5

Superposition skip

Green's function solution can be found for inhomogeneous linear equations.

inhomogeneous $\rightarrow$ there is a source term

linear $\rightarrow$ the operator acting on the unknown function ϕ is linear

$$L(\phi_1+\phi_2) = L\phi_1 + L\phi_2$$

in our case $L = \nabla^2$

we can write the ~~answer~~ solution

$$\phi(\underset{\sim}{x}) = \int d^3x' \, G(x,x') \, \rho(x')$$

$G(x,x')$ gives the value of ϕ at point x (response) due to charge density at x' (source)

~~For One more tp~~

for the solutions of poisson's we have discussed so far

$$G(x,x') = \frac{1}{|x-x'|}$$

and $G(x,x')$ satisfies

$$\boxed{\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')}$$

NOTE that the Green's function satisfies ~~Laplaces~~ Poisson's equation with a delta function source at the point x'.

~~So f~~ boundary condit.

$$G \to 0 \quad \text{as} \quad |x| \to \infty$$

One thing that we have not discussed is ~~the~~ boundary conditions. The solution

$$\phi(\underset{\sim}{x}) = \int \frac{d^3x' \, \rho(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

applies if all charge density is included in $\rho(\underset{\sim}{x})$ and it is assumed that

$$\phi(\underset{\sim}{x}) \to 0 \quad \text{as } |\underset{\sim}{x}| \to \infty$$

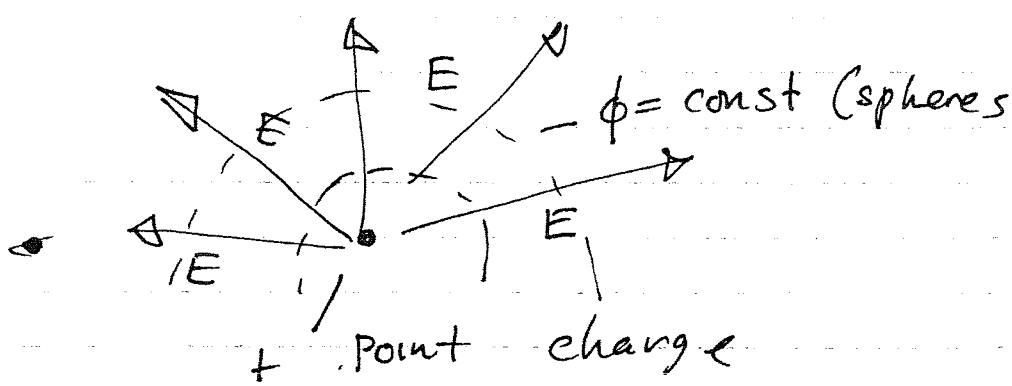
if $\rho(\underset{\sim}{x}')$ ~~is~~ occupies a finite volume, then as $|\underset{\sim}{x}| \to \infty$

$$\phi(\underset{\sim}{x}) \simeq \frac{1}{|\underset{\sim}{x}|} \int d^3x' \rho(\underset{\sim}{x}') \to 0$$

as $|\underset{\sim}{x}| \to \infty$

We have already seen that with conductors present ~~char~~ surface charge density appears on the conductors so that $\underset{\sim}{E} = 0$ in the conductor. The form of this surface charge density is not known apriori but must be determined ~~along~~ as part of the solution

1 Schematic Example



Uniqueness of solutions

consider

$$\nabla^2 \phi = - \cancel{4\pi}\rho/\epsilon_0$$

subject to boundary conditions on enclosing ~~sur~~ surface

Suppose we have found a solution satisfying the equation and all B.C's. Is that solution unique? ans. yes

Proof by contradiction

suppose ϕ_1 and ϕ_2 are two different solutions then $U = \phi_1 - \phi_2$ satisfies

$$\nabla^2 u = 0$$

with B.C's $\left\{ \begin{array}{l} U = 0 \\ \text{if } \phi \text{ satisfies} \\ \text{Dirichlet} \\ \\ \dfrac{\partial U}{\partial n} = 0 \text{ if } \phi \\ \text{satisfies Neumann} \\ \text{B.C.} \end{array} \right.$

$$\int d^3x\, U \underbrace{\nabla^2 U}_{0} = \int_V d^3x \left[(\nabla \cdot U \nabla U) - |\nabla U|^2 \right] = 0$$

THUS $\displaystyle \int_V d^3x\, |\nabla U|^2 = \int_S da\, (\mathbf{n} \cdot \nabla U)\, U$

if either U or ~~$\partial U/\partial n$~~ $n \cdot \nabla U$ $= 0$ on S then we must have

$$\int_V d^3x\, |\nabla U|^2 = 0$$

$U = \text{const}$

for dirichlet $U = 0$

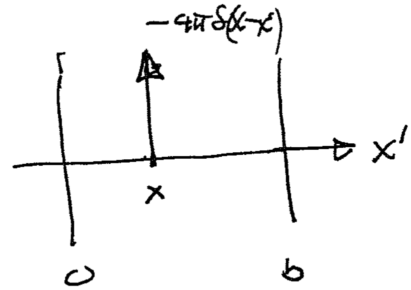
for neumann constant is un important

From this it follows that we can't specify both ϕ and $n \cdot \nabla \phi$

Example 1D

$$\frac{d^2}{dx'^2} G_D(x,x') = -4\pi\delta(x'-x)$$

sheet charge



$$G_D(x,0) = G_D(x,b) = 0$$

for $0 < x' < x$

$$G_D = K_1 x'$$

here K_1, K_2 are independent of x'

for $x < x' < b$ $\quad G_D = K_2(x'-b)$

at $x' = x$ $\quad G_D$ is continuous

$$K_1 x = K_2(x-b)$$

$$\int_{x-\epsilon}^{x+\epsilon} dx' \frac{d^2}{dx'^2} G_D = \frac{d}{dx'} G_D \Big|_{x-\epsilon}^{x+\epsilon} = -4\pi$$

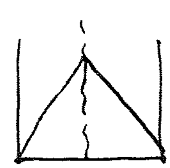
$$K_2 - K_1 = -4\pi \qquad K_1 = K_2 + 4\pi$$

$$\cancel{K_2 + K_2 =} \quad (K_2 + 4\pi) = K_2\left(1 - \frac{b}{x}\right)$$

$$\cancel{K_2 = \frac{b}{4\pi x} \; 2K_2 =} \qquad K_2 = -4\pi\frac{x}{b}$$

$$K_1 = 4\pi\left(1 - \frac{x}{b}\right)$$

$$G_D = \begin{cases} 4\pi\left(1 - \frac{x}{b}\right)x' & 0 < x' < x \\ -4\pi\frac{x}{b}(x'-b) & x < x' < b \end{cases}$$



CHAPTER 6

Variational Principles

Consider the following variational problem

$$I[\psi] = \int_V d^3x \left[\frac{|\nabla\psi|^2}{2} - \frac{\rho(\underset{\sim}{x})}{\epsilon_0}\psi \right]$$

where $\rho(\underset{\sim}{x})$ is a specified function in V and $\psi(\underset{\sim}{x})$ is an arbitrary ~~func~~ continuous function of $\underset{\sim}{x}$ subject to ~~boundary~~ conditions $\psi(\underline{x}) = \phi_B(\underline{x})$ on boundary of V

$I[\psi]$ is minimized when

$\psi = \phi(\underline{x})$ solution of

$\nabla^2\phi(x) = -\rho(x)/\epsilon_0 \qquad \phi(x) = \phi_B$ on boundary

MINIMIZATION

Show:

Let $\psi(x) = \psi_m(x) + \epsilon\,\psi_T(x)$

$\psi_m(x)$ is that function that minimizes I
$\delta\psi(x)$ is a deviation that is arbitrary except $\delta\psi(x) = 0$ on boundary

$$I = \int d^3x \left[\frac{|\nabla\psi_m|^2}{2} - \frac{\rho(x)}{\epsilon_0}\psi_m \right] + \int d^3x\, \frac{\epsilon^2 |\nabla\psi_T|^2}{2}$$

$$+ \epsilon \int d^3x \left[\nabla\psi_T \cdot \nabla\psi_m - \frac{\rho}{\epsilon_0}\delta\psi_T \right]$$

I will be an extremum if term proportional to $\delta\psi$ vanishes

should be true for any $\delta\psi_T$

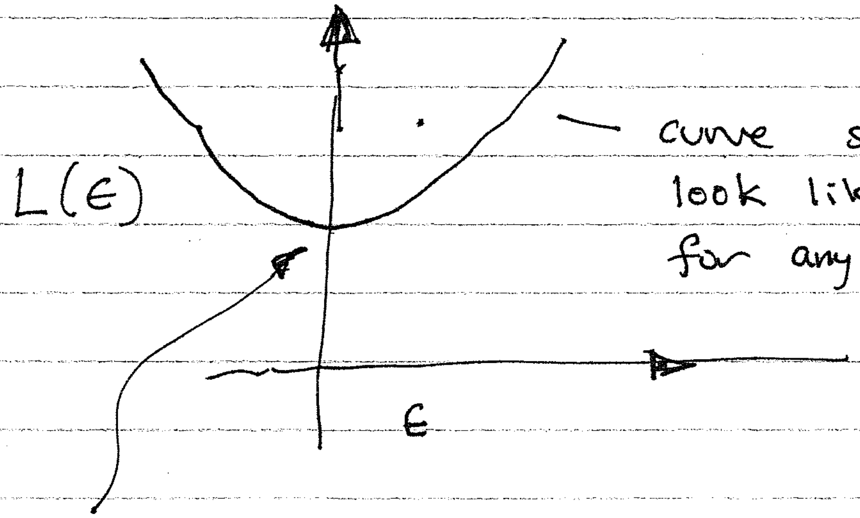
$$\int d^3x \left[\nabla\psi_T \cdot \nabla\psi_m - \frac{\rho}{\epsilon_0}\delta\psi_m \right] = -\int d^3x\, \delta\psi_T \underbrace{\left[\nabla^2\psi_m + \frac{\rho}{\epsilon_0} \right]}_{0}$$

$$+ \int_B da\, \delta\psi (n \cdot \nabla\psi_m) = 0$$

$\delta\psi = 0$ on B

$$\frac{dL}{d\epsilon} = \int_{x_a}^{x_b} dx \, \frac{1}{\sqrt{1+\left(\frac{dy_0}{dx}+\epsilon\frac{dy_t}{dx}\right)^2}}$$

$$\left(\frac{dy_0}{dx}+\epsilon\frac{dy_t}{dx}\right)\frac{dy_t}{dx}$$



curve should look like this for any ~~dy_t/d~~ $y_t(x)$

$$\left.\frac{dL}{d\epsilon}\right|_{\epsilon} = 0$$

let $\epsilon \to 0$

$$\frac{dL}{d\epsilon} = \int_{x_a}^{x_b} dx \, \frac{dy_t}{dx}\left[\frac{dy_0}{dx}\frac{1}{\sqrt{1+\left(\frac{dy_0}{dx}\right)^2}}\right] = 0$$

$$0 = \left.\frac{dL}{d\epsilon}\right|_{\epsilon=0} = y_t \left[\frac{y_0'}{\sqrt{1+y_0'^2}}\right]\Bigg|_{x_a}^{x_b} - \int_{x_a}^{x_b} dx\, y_t \frac{d}{dx}\frac{y_0'}{\sqrt{1+y_0'^2}}$$

$\cdot\ y_t(x_b) = y_t(x_a) = 0$

$$\int_{x_b}^{x_b} dx\ y_t\ \frac{d}{dx}\left(\frac{y_0'}{\sqrt{1+y_0'^2}}\right) = 0$$

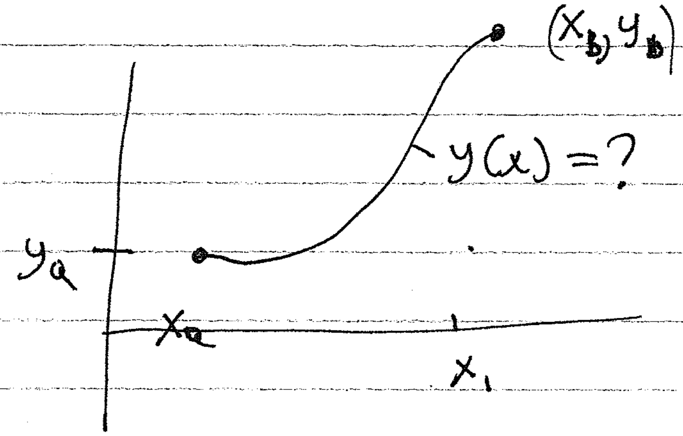
for any $y_t(x)$!

Requires $\frac{d}{dx}\frac{y_0'}{\sqrt{1+y_0'^2}} = 0$

$$\boxed{y_0' = \text{const}}$$

Calculus of Variations

Show that a straight line gives the path with minimum ~~arc~~ arc length



constraints

$y_a = y(x_a)$

$y_b = y(x_b)$

$$L = \int_{x_a}^{x_b} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

actual solution

test solution

let $y(x) = y_0 + \epsilon y_t$

$$L = \int_{x_a}^{x_b} dx \sqrt{1 + \left(\frac{dy_0}{dx} + \epsilon \frac{dy_t}{dx}\right)^2}$$

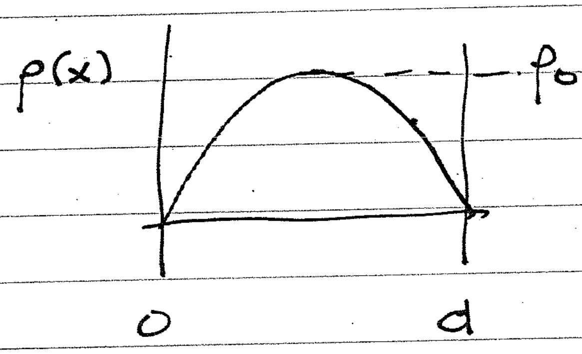
$y_t(x_a) = y_t(x_b) = 0$

1D Example:

$$I[\psi] = \int_0^d dx \left[\frac{1}{2}\left(\frac{d\psi}{dx}\right)^2 - \frac{\rho(x)}{\epsilon_0}\psi(x) \right]$$

PROBLEM $\quad \psi(0) = \psi(d) = 0$

$$\rho(x) = \rho_0 \sin \frac{\pi x}{d}$$



$\boxed{\text{actual solution : } \cancel{\rho(x) =}}$

$$\phi(x) = \phi_0 \sin\frac{\pi x}{d} \qquad \phi'' = -\frac{\pi^2}{d^2}\phi_0 \sin\frac{\pi x}{d}$$

$$\frac{\pi^2}{d^2}\phi_0 = \frac{\rho_0}{\epsilon_0} \qquad \boxed{\phi_0 = \frac{d^2}{\pi^2}\frac{\rho_0}{\epsilon_0}}$$

④

$$\frac{d\phi}{dx} = \frac{4\phi_0}{d^2}\left[d-2x\right]$$

$$\int_0^d dx\, \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = \frac{8}{3}\,\frac{\phi_0^2}{d^2}\,d$$

~~Compare with exact~~

$$\int_0^d dx\, \frac{\rho_0}{\epsilon_0}\sin\frac{\pi x}{d}\,\frac{4\phi_0}{d^2}\,x(d-x) = \frac{\rho_0}{\epsilon_0}\,\phi_0\,\frac{16 d}{\pi^3}$$

TRIAL

$$I(\phi_0) = \frac{d}{2}\left[\underbrace{\frac{16}{3}}_{5.33}\,\frac{\phi_0^2}{d^2} - \underbrace{\frac{32}{\pi^3}}_{1.032}\,\frac{\rho_0}{\epsilon_0}\,\phi_0\right]$$

EXACT

$$I(\phi_0) = \frac{d}{2}\left[\overbrace{\frac{\pi^2}{2}}^{4.935}\,\frac{\phi_0^2}{d^2} - \overbrace{\frac{\rho_0\phi_0}{\epsilon_0}}^{1}\right]$$

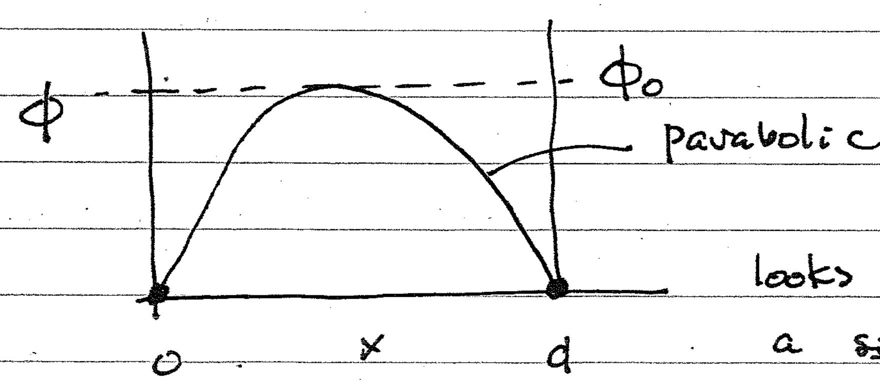
$$\therefore \quad \phi_0 = \frac{d^2}{\pi^2} \frac{\rho_0}{\epsilon_0}$$

In this case we get the exact result because of our lucky guess

$$I_{min} = -\frac{d}{2}\left[\frac{1}{2}\left(\frac{\rho_0}{\epsilon_0}\right)^2 \frac{d^2}{\pi^2}\right]$$

Now try a good but not great guess

$$\phi = \frac{4\phi_0}{d^2} x(d-x)$$



looks like a ~~sign wav~~ sine

Minimize

$$\frac{32}{3}\frac{\Phi_0}{d^2} - \frac{32}{\pi^3}\frac{\rho_0}{\epsilon_0} = 0$$

EXACT RESULT

$$\Phi_0 = \frac{3}{\pi}\left(\frac{d^2}{\pi^2}\frac{\rho_0}{\epsilon_0}\right)$$

0.95

TRIAL

$$I_{min} = \frac{d}{2}\left[\frac{16}{3}\cdot\frac{1}{d^2}\frac{3^2}{\pi^2}\left(\frac{d^2\rho_0}{\pi^2\epsilon_0}\right)^2 - \frac{32}{\pi^3}\frac{\rho_0}{\epsilon_0}\frac{3}{\pi}\left(\frac{d^2\rho_0}{\pi^2\epsilon_0}\right)\right]$$

$$= \frac{d}{2}\left[\left(\frac{\rho_0}{\epsilon_0}\right)^2\frac{d^2}{\pi^2}\right]\left(\frac{32}{\pi^3}\frac{3}{\pi} - \frac{16\cdot 3}{\cancel{3}\,\pi^4}\right) = -0.4928$$

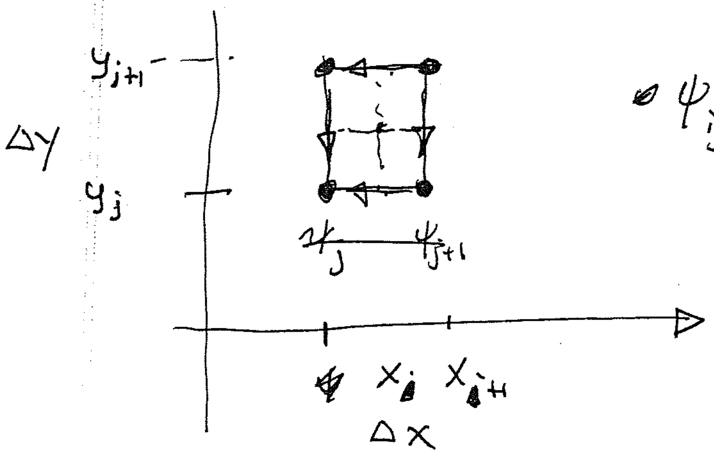
$$\frac{16\cdot 3}{\cancel{3}\,\pi^4} = 0.4928$$

vs $= 0.5$

$$I_{min}^{TRIAL} > I_{min}^{exact}$$

CHAPTER 7

More Practical Example   Numerical Method



ψ_{ij} defined on 2D grid

$$\frac{1}{2}\int d^2x\, |\nabla\psi|^2 \Rightarrow \frac{1}{2}\sum_{\substack{\text{Squares}\\(i,j)}} \Delta x\,\Delta y \left\{ \left|\frac{1}{2\Delta x}\left[(\psi_{i+1,j+1}+\psi_{i+1,j}) - (\psi_{i,j+1}+\psi_{i,j})\right]\right|^2 + \left[\frac{1}{2\Delta y}\left[(\psi_{i+1,j+1}+\psi_{i,j+1}) - (\psi_{i+1,j}+\psi_{ij})\right]\right]^2 \right\}$$

$$\int d^3x\, \frac{\rho}{\epsilon_0}\psi \Rightarrow \sum_{\text{Square}} \Delta x\,\Delta y\, \frac{\rho_{ij}\psi_{ij}}{\epsilon_0}$$

$I = I\{\psi_{ij}\}$ minimize numerically
converges to solution