CHAPTER 4
Boundary Conditions on Surfaces

Consider

\[ \phi(x) = \int \frac{d^3x'}{4\pi\varepsilon_0 |x-x'|} \rho(x') \]

\[ E = -\nabla \phi(x) \Rightarrow \oint E \cdot d\ell = 0 \]

represents a complete solution for the electric field given the charge density. Problem is we do not always know the charge density beforehand, but we must sometimes determine it as part of the solution of the problem.

To do this we then require...
For example suppose we introduce an electrical conductor into our problem. Charge on such a conductor is free to move and arrange itself such that \( E = 0 \) within the conductor. Once \( E = 0 \) there no longer will be any electrical force on a free charge within the conductor inducing the charge to move. (This does not imply that there is no force on the conductor as we shall see.)
In conductor \( \nabla \cdot \vec{E} = \nabla \cdot \vec{Q} = 0 \)

no net charge density inside conductor

At boundary

\[ \oint \vec{E} \cdot d\vec{l} = 0 \]

closed loop

This implies \( \vec{E}_{tv} - \vec{E}_{tc} = 0 \)

but \( \vec{E}_{tc} = 0 \) (\( \vec{E}_0 \) in cond.)
thus \( E_{tr} = 0 \) (two components)

\[
\phi(1) - \phi(2) = - \int_{C} \mathbf{E} \cdot d\mathbf{l}
\]

taking path to lie in the surface implies that \( \phi = \text{const} \)
on surface of conductor.

this will be used to determine \( \text{Normal Component} \)

\[ E_{\text{normal}} = \text{component} \]

\[ E_{\text{normal}} \]

\[ E_0 \int \mathbf{d}a \cdot \mathbf{N} \cdot \mathbf{E} = 4\pi \frac{q_{\text{enclosed}}}{E_0} \]

\( E = 0 \) inside

\( h \to 0 \) \( \mathbf{N} \cdot \mathbf{E} = 4\pi \left( \frac{q_{\text{enclosed}}}{E_0 \mathbf{d}a} \right) \)

surface charge density
$\sigma = \text{surface charge density}$

Thus, $\delta$

In general, there is a discontinuity in the normal component of $E$ field at a surface with surface charge density.

$\kappa (E_1 - E_2)_{\text{tangent}} = 0$

Normal partition from 2 to 1

$(E_1 - E_2) \cdot n = \frac{\text{surf} \sigma}{\varepsilon_0}$

$x$ for surface charge density at $E_2 = 0$
Potential due to surface charge density

\[ \phi(x) = \int \frac{\sigma(x')}{|x-x'|} \, dx' \]

**Dipole**  
Continuous

Other special charge distributions

Other distributions dipole we will come to this,

Caution while above is true you almost never know \( \sigma(x') \) a priori
suppose we move a perfect conductor into the picture

free charge within conductor arranges itself so that $E = 0$ inside conductor

Thus what we want to solve is

$$\varepsilon_0 \nabla^2 \phi = -\frac{q}{\varepsilon_0} \rho(x) \quad \text{in vacuum}$$

with $\phi = \text{const}$ on conductor

total charge on
Consider

Thus, the kind of problem we want to solve is

\[ \varepsilon_0 \nabla^2 \phi = - \frac{\partial \rho}{\partial t}(x) \]

does not include surface charge on boundary

along with certain boundary conditions on \( \phi \) or at normal derivative on given surfaces

\( \phi \) specified on surface

"Dirichlet" (usual case if conductors) present

\[ \mathbf{n} \cdot \nabla \phi = \frac{\partial \phi}{\partial n} \] specified "Neumann"

if surface charge

Caution

For Poisson can not specify

both on all surfaces

switch to unspecified
What boundary conditions?

\[ \phi = \text{cons potential held fixed} \]

\[ \oint_{\text{spec.}} \mathbf{a} \cdot \nabla \phi = 0 \quad (\text{no net surface charge}) \]

\[ \oint_{\text{fixed charge on sphere}} \]
CONSIDER POISSON'S EQUATION

\[ \varepsilon_0 \nabla^2 \phi = - \rho(x) \]

which has the solution

\[ \phi(x) = \int d^3 x' \frac{\rho(x')}{4\pi\varepsilon_0 |x-x'|} \]

THIS MEANS

\[ \nabla^2 \phi(x) = \int d^3 x' \frac{\rho(x')}{4\pi\varepsilon_0} \nabla^2 \frac{1}{|x-x'|} \]

\[ = - \frac{4\pi\rho(x)}{\varepsilon_0} \]
\[ \rho(x) = \int d^3x' \rho(x') \left[ -\frac{1}{4\pi} \frac{\nabla^2}{|x-x'|} \right] \]

A very special function

\[ \delta(x-x') = -\frac{1}{4\pi} \frac{\nabla^2}{|x-x'|} \]

true for any \( \rho(x) \)

\[ \delta(x-x') = \begin{cases} 0 & \text{if } x \neq x' \\ \infty & \text{if } x = x' \end{cases} \]

Dirac delta function

\[ \int d^3x' \delta(x-x') = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \]

but \[ \int d^3x \delta(x) = 1 \]
TWO USEFUL RELATIONS

\[ \nabla \frac{1}{|x-x'|} = \frac{(x-x')}{|x-x'|^3} \]

\[ \nabla \cdot \nabla \frac{1}{|x-x'|} = \nabla \cdot \left( \frac{-(x-x')}{|x-x'|^3} \right) = -4\pi \delta (x-x') \]
\[ \text{Property of \( \delta \)-delta functions:} \]
\[ \int_{-\infty}^{\infty} f(x) \delta(x - a) \, dx = f(a) \]

where \( f(x) \) is a function.

\[ \rho(x) = \sum_{i=1}^{n} \delta(x - x_i^A) \]

\[ \rho(x) = \frac{1}{\pi} \frac{e^{x^2}}{x} \]

\[ \text{Localized to express pt charge at various charges} \]

\[ \text{Dirac Delta Functions} \]

\[ \text{Volume charge density} \]

\[ \rho(x) = \frac{1}{\pi} \frac{e^{x^2}}{x} \]

\[ \int_{-\infty}^{\infty} \delta(x - x_i^A) \, dx = \delta(x - x_i^A) \]

\[ \text{Field strength in that point} \]
(i) $S(x-a) = 0$ if $x \neq a$.

(ii) $\int_{c}^{d} S(x-a) \, dx = \frac{1}{a}$ if $b < a < c$ and undefined otherwise.

(iii) $\int_{-\infty}^{a} S(x-a) \, dx = f(a)$ (assume $f(x)$ is continuous at $x = a$).

(iv) $\int_{-\infty}^{+\infty} f(x) S(x-a) \, dx = -f'(a)$ (by parts)

(v) $f'(x) = df/dx$

\[ g(x) \text{ has 3 zeros (n=3)} \]

\[ S(g(x)) = \sum_{i=1}^{n} \frac{S(x-x_i)}{|g'(x_i)|} \]

(vi) $S(cx) = \frac{1}{|c|} S(x)$. (from (vi) $\int_{-\infty}^{+\infty} S(cx) \, dx = \frac{1}{|c|}$)

Derivation of (vi)

\[ \int_{-\infty}^{+\infty} S(g(x)) \, dx = \int_{-\infty}^{+\infty} S(E(x-x_i)g'(x_i)) \, dx = \frac{1}{|c|} \]
Example: Represent the three dimensional delta function $\delta(x-x')$ in cylindrical coordinates.

\[
\delta(x-x') = \frac{\delta(p-p') \delta(\theta-\theta') \delta(z-z')}{p'}
\]

\[
d^3x' = \rho' d\rho' d\theta' dz'
\]

So, \[ \int_V \delta(x-x') d^3x' = 1 \] if \((p', \theta', z')\) is in \(V\).

In general, under metric coefficients, \((u_1, u_2, u_3)\): orthogonal coordinate system

\[
d^3x = h_1 du_1 du_2 du_3
\]

\[
\delta(x-x') = \frac{\delta(u_1-u_1') \delta(u_2-u_2') \delta(u_3-u_3')}{h_1 h_2 h_3}
\]

Example in cylindrical coordinates,

\[
u_1 = \rho, \quad u_2 = \theta, \quad u_3 = z
\]

\[
h_1 = \rho, \quad h_2 = 1, \quad h_3 = \text{constant}
\]

Spherical coordinates,

\[
u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi
\]

\[
dV = r^2 \sin \phi dr d\theta d\phi
\]
Green's Theorem and Green's Function

Consider Poisson's equation

\[ \varepsilon_0 \nabla^2 \phi = -4\pi \rho(x) \]

and its "solution"

\[ \phi(x) = \int d^3 x' \frac{\rho(x')}{4\pi\varepsilon_0 |x - x'|} \]

This is a form of a Green's function solution.
TRUE FOR ANY CHOICE OF $\phi, \psi$

Now let $\phi$ be the solution to the problem we are interested in:

$$-\nabla^2 \phi = \frac{P(x)}{\varepsilon} \quad \text{(haven't specified BC's)}$$

Let $\psi(x)$ be the solution to

$$\nabla^2 \psi = -4\pi \delta(x-x') \quad \text{(haven't specified BC's)}$$

(example $\psi = \frac{1}{|x-x'|}$)

$$\int_V d^3x \left[ \phi(x)(-4\pi \delta(x-x')) + \psi(x,x') \frac{P(x)}{\varepsilon_0} \right]$$

$$= \int_S d\alpha \left[ \phi n \cdot \nabla \psi - \psi n \cdot \nabla \phi \right]$$

**FIRST TERM** = $-4\pi \phi(x')$ if $x'$ is in $V$

= 0 otherwise
\[ \phi(x') = \int \frac{d^3x \psi(x)}{4\pi \epsilon_0} \psi(x, x') \]

\[ -\frac{1}{4\pi} \int_S da [\phi \nabla \psi - \psi \nabla \phi] \]

**Boundary Conditions**

(Dirichlet problem)

Suppose \( \phi(x) \) is known on boundary

(then \( n \cdot \nabla \phi \) is unknown)

**Pick** \( \psi \equiv G(x, x') \)

so that \( \frac{\partial G}{\partial x} = 0 \)

at \( x = \text{boundary} \)

Call \( G_0 \)

\[ \phi(x') = \int \frac{d^3x \rho(x)}{4\pi \epsilon_0} G(x, x') \]

\[ -\frac{1}{4\pi} \int_S da \phi(x) \nabla G(x, x') \]

\( \phi = 0 \) at boundary

\(-\) Homogeneous term from potential on boundary
Symmetry of

\[ + \nabla^2 G_D = -4\pi \delta(x-x') \]

\[ G_D (x, x') \bigg|_{x = \text{Boundary}} = 0 \]

\[-\nabla^2 \phi = \frac{\rho(x)}{\varepsilon_0} \quad \phi = 0 \quad \text{on} \ B\]

\[ \phi(x') = \frac{1}{4\pi\varepsilon_0} \int_V d^3x \rho(x) G_D(x, x') \]

Now suppose \( \rho(x) = 4\pi\varepsilon_0 \delta(x-x'') \)

Then \( \phi(x) = G_D(x, x'') \)

\[ \phi(x') = G_D(x'', x') \]

\[ \phi(x) = G_D(x'', x) \]

arguments reversed (reciprocity)
Green's Theorem

Follows from divergence theorem

\[
\int_V \nabla \cdot \mathbf{A} = \int_S \mathbf{A} \cdot n \, da
\]

Let \( \mathbf{A} = \phi \nabla \psi \)

\( \phi, \psi \) are two arbitrary scalars

\[ \nabla \cdot \mathbf{A} = \nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \]

product rule

\[
\int_V [\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi] \, dx = \int_S \phi \mathbf{n} \cdot \nabla \psi \, da
\]

interchange \( \phi \leftrightarrow \psi \)

\[
\int_V (\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi) \, dx = \int_S \psi \mathbf{n} \cdot \nabla \phi \, da
\]

\[
\int_V [\phi \nabla^2 \psi - \psi \nabla^2 \phi] \, dx = \int_S [\phi \mathbf{n} \cdot \nabla \psi - \psi \mathbf{n} \cdot \nabla \phi] \, da
\]
Green's function solution can be found for inhomogeneous linear equations.

**inhomogeneous** → there is a source term

**linear** → the operator acting on the unknown function $\phi$ is linear

$$L(\phi_1 + \phi_2) = L\phi_1 + L\phi_2$$

In our case, $L = \nabla^2$

We can write the answer:

$$\phi(x) = \int d^3x' G(x,x') \rho(x')$$

$G(x,x')$ gives the value of $\phi$ at point $x$ due to the response of $\rho$.
charge density at $x'$ (source)

for the solutions we have discussed so far

$$G(x, x') = \frac{1}{|x-x'|}$$

and $G(x, x')$ satisfies

$$\nabla^2 G(x, x') = -4\pi \delta(x-x')$$

Note that the Green's function satisfies Laplace's Poisson's equation with a delta function source at the point $x'$.

So, boundary condition:

$$G \to 0 \text{ as } |x| \to \infty$$
One thing that we have not discussed is the boundary conditions. The solution

\[ \phi(x) = \int d^3 x' \frac{\rho(x')}{|x - x'|} \]

applies if all charge density is included in \( \rho(x) \) and it is assumed that

\[ \phi(x) \to 0 \quad \text{as} \quad |x| \to \infty \]

if \( \rho(x') \) occupies a finite volume, then as \( |x| \to \infty \)

\[ \phi(x) \approx \frac{1}{|x|} \int d^3 x' \rho(x') \to 0 \quad \text{as} \quad |x| \to \infty \]
We have already seen that with conductors present a surface charge density appears on the conductors so that $\mathbf{E} = 0$ in the conductor. The form of this surface charge density is not known a priori but must be determined along as part of the solution.

\underline{Schematic} \hspace{1cm} \underline{Example}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{schematic.png}
\caption{Example of a spherical charge distribution}
\end{figure}
Uniqueness of solutions

Consider

$$\nabla^2 \phi = -\frac{q_0}{\epsilon_0}$$

subject to boundary conditions on enclosing surface

Suppose we have found a solution satisfying the equation and all BC's. Is that solution unique? Ans. yes

Proof by contradiction

Suppose $\phi_1$ and $\phi_2$ are two different solutions then $U = \phi_1 - \phi_2$ satisfies

$$\nabla^2 U = 0$$

with BC's

$$\begin{cases} 
U = 0 & \text{if } \phi \text{ satisfies Dirichlet} \\
\frac{\partial U}{\partial n} = 0 & \text{if } \phi \text{ Satisfies Neumann BC}
\end{cases}$$
\[ \int d^3x \nabla^2 u = \int d^3x [\nabla \cdot (u \nabla u)] - 4 \nabla u \cdot \nabla u = 0 \]

Thus \[ \int d^3x |\nabla u|^2 = \int_S (n \cdot \nabla u) u \]

If either \( u \) or \( \nabla \cdot (u \nabla u) = 0 \) on \( S \) then we must have

\[ \int d^3x |\nabla u|^2 = 0 \]

\[ u = \text{const} \]

for Dirichlet \( u = 0 \)

for Neumann \( \text{constant is an important} \)

From this it follows that we can't specify both \( f \) and \( a \).
Example 1D

\[ \frac{d^2}{dx^2} G_D(x, x') = -4\pi \delta(x-x') \]

\[ G_D(x, 0) = G_D(x, b) = 0 \]

for \( 0 < x' < x \)

\[ G_D = K_1 x' \]

for \( x < x' < b \)

\[ G_D = K_2 (x'-b) \]

at \( x' = x \) \( G_D \) is continuous

\[ K_1 x = K_2 (x-b) \]

\[ \int_{x-e}^{x+e} \frac{d^2}{dx'^2} G_D \, dx' = \frac{d}{dx'} G_D \bigg|_{x-e}^{x+e} = -4\pi \]

\[ K_2 - K_1 = -4\pi \]

\[ K_1 = K_2 + 4\pi \]

\[ K_2 = -4\pi \frac{x}{b} \]

\[ K_1 = 4\pi (1 - \frac{x}{b}) \]

\[ G_D = \begin{cases} 
4\pi (1 - \frac{x}{b}) x' & 0 < x' < x \\
-4\pi \frac{x}{b} (x'-b) & x < x' < b 
\end{cases} \]
Variational Principles

Consider the following variational problem

\[ I_\Psi [\Psi] = \int_\Omega \left[ \left| \nabla \Psi \right|^2 - \frac{\rho(x)\Psi}{\varepsilon_0} \right] \, d^3x \]

where \( \rho(x) \) is a specified function in \( \Omega \) and \( \Psi(x) \) is an arbitrary continuous function of \( x \) subject to boundary conditions \( \Psi(x) = \phi_b(x) \) on boundary of \( \Omega \).

\[ I_\Psi [\Psi] \] is minimized when

\[ \Psi = \phi(x) \] solution of

\[ \nabla^2 \phi(x) = -\frac{\rho(x)}{\varepsilon_0} \quad \phi(x) = \phi_b \] on boundary

MINIMIZATION
Show:

Let \( \psi(x) = \phi_m(x) + e^{i I/\hbar} \psi(x) \)

\( \phi_m(x) \) is the function that minimizes \( I \)
\( \psi(x) \) is a deviation that is arbitrary except \( \psi(x) = 0 \) on boundary

\[
I = \int d^3 x \left[ \frac{\left| \nabla \phi_m \right|^2}{2} - \frac{\rho(x) \phi_m}{\varepsilon_0} \right] + \int d^3 x \frac{e^2}{2} \left| \nabla \psi \right|^2
\]

\[
+ \int d^3 x \left[ \nabla \phi_T \cdot \nabla \phi_m - \frac{e \phi_T}{\varepsilon_0} \phi_m \right]
\]

\( I \) will be an extremum if term proportional to \( \psi \) vanishes

\[
\int d^3 x \left[ \nabla \phi_T \cdot \nabla \phi_m - \frac{e \phi_T}{\varepsilon_0} \phi_m \right] = -\int d^3 x \frac{\phi_T}{\varepsilon_0} \left[ \nabla^2 \phi_m + \frac{\rho}{\varepsilon_0} \right]
\]

\[
+ \int d^3 x \phi_T \nabla \phi_m \cdot \nabla \phi_m = 0
\]

\( \psi = 0 \) on \( \partial \Omega \)
\[ \frac{dL}{d\varepsilon} = \int_{x_a}^{x_b} dx \left[ \frac{1}{\sqrt{1 + \left( \frac{dy_0}{dx} + \varepsilon \frac{dy_1}{dx} \right)^2}} \right] \]

\[ = \left( \frac{dy_0}{dx} + \varepsilon \frac{dy_1}{dx} \right) \frac{dy_1}{dx} dx \]

\[ L(\varepsilon) \]

\[ \text{curve should look like this for any } y_0(x), y_1(x) \]

\[ \frac{dL}{d\varepsilon} \bigg|_{\varepsilon=0} = 0 \]

\[ \text{let } \varepsilon \to 0 \]

\[ \frac{dL}{d\varepsilon} = \int_{x_a}^{x_b} dx \frac{dy_1}{dx} \left[ \frac{1}{\sqrt{1 + \left( \frac{dy_0}{dx} \right)^2}} \right]^2 = 0 \]
\[ 0 = \left. \frac{dL}{d\epsilon} \right|_{\epsilon=0} = y_t \left[ \frac{y_0'}{\sqrt{1 + y_0'^2}} \right] \bigg|_{x_a}^{x_b} - \int_{x_a}^{x_b} d\epsilon \ y_t \frac{d}{d\epsilon} \left( \frac{y_0'}{\sqrt{1 + y_0'^2}} \right) \]

\[ y_t(x_0) = y_t(x_0) = 0 \]

\[ \int_{x_a}^{x_b} d\epsilon \ y_t \frac{d}{d\epsilon} \left( \frac{y_0'}{\sqrt{1 + y_0'^2}} \right) = 0 \]

for any \( y_t(x) \)!

Requires

\[ \frac{d}{d\epsilon} \left( \frac{y_0'}{\sqrt{1 + y_0'^2}} \right) = 0 \]

\[ y_0'^2 = \text{constant} \]
Show that a straight line gives the path with minimum arc length.

\[ y(x) = \begin{cases} y_a & \text{if } x = x_a \\ y_b & \text{if } x = x_b \end{cases} \]

Constraints:
\[ y_a = y(x_a) \]
\[ y_b = y(x_b) \]

\[ L = \int_{x_a}^{x_b} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

Actual solution:
\[ y(x) = y_0 + \varepsilon y_1 \]

Test solution:
\[ L = \int_{x_a}^{x_b} \sqrt{1 + \left( \frac{dy_0}{dx} + \varepsilon \frac{dy_1}{dx} \right)^2} \, dx \]

\[ y_1(x_a) = y_1(x_b) = 0 \]
1D Example:

\[
\int_{\psi} = \int_0^d dx \left[ \frac{1}{2} \left( \frac{d\psi}{dx} \right)^2 - \frac{p(x)}{\varepsilon_0} \psi(x) \right]
\]

Problem \quad \psi(0) = \psi(d) = 0

\[p(x) = p_0 \sin \frac{\pi x}{d}\]

\[\phi(x) = \phi_0 \sin \frac{\pi x}{d}, \quad \phi'' = -\frac{\pi^2}{d^2} \phi_0 \sin \frac{\pi x}{d}\]

\[\frac{\pi^2}{d^2} \phi_0 = \frac{p_0}{\varepsilon_0}, \quad \phi_0 = \frac{d^2 p_0}{\pi^2 \varepsilon_0}\]
\[ \frac{d\phi}{dx} = \frac{4\phi_0}{d^2} \left[ d - 2\phi_0 x \right] \]

\[ \int_{0}^{d} dx \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 = \frac{8}{3} \frac{\phi_0^2}{d} \]

Compare with exact

\[ \int_{0}^{d} dx \frac{\rho_o}{\varepsilon_o} \sin \frac{\pi x}{d} \frac{4\phi_0}{d^2} x (d-x) = \frac{\rho_o}{\varepsilon_o} \frac{\phi_0}{60} \frac{16d}{\pi^3} \]

**TRIAL**

\[ I(\phi_0) = \frac{d}{2} \left[ \frac{4}{3} \frac{\phi_0^2}{d^2} - \frac{32}{\pi^3} \frac{\rho_o}{\varepsilon_o} \phi_0 \right] \]

\[ d = 4.533 \]

**Exact**

\[ I(\phi_0) = \frac{d}{2} \left[ \frac{\pi^2}{2} \frac{\phi_0^2}{d^2} - \frac{\rho_o \phi_0}{\varepsilon_o} \right] \]

\[ d = 4.935 \]
\[ \phi_0 = \frac{d^2 \rho_0}{\pi^2 c_0} \]

In this case we get the exact result because of our lucky guess

\[ I_{\text{min}} = -\frac{d}{2} \left[ \frac{1}{(\rho_0)}^2 \frac{d^2}{\pi^2} \right] \]

Now try a good but not great guess

\[ \phi = 4 \frac{\phi_0}{d} x (d-x) \]

\[ \phi \]

\[ \phi_0 \]

Parabolic looks like a sine
Minimise

\[ \frac{32}{3} \Phi_0 - \frac{32}{\pi^3} \rho_0 = 0 \]

\[ \Phi_0 = \frac{3}{\pi} \left( \frac{d^2 \rho_0}{\pi^2 \epsilon_0} \right) \]

Exact Result

\[ 0.95 \]

Trial

\[ I_{min} = \frac{d}{2} \left[ \frac{16.1}{3} \frac{3^2}{\pi^2} \left( \frac{d^2 \rho_0}{\pi^2 \epsilon_0} \right)^2 - \frac{32}{\pi^3} \rho_0 \frac{3}{\pi} \left( \frac{d^2 \rho_0}{\pi^2 \epsilon_0} \right) \right] \]

\[ \approx \frac{d}{2} \left[ \left( \frac{\rho_0}{\epsilon_0} \right)^2 \frac{d^2}{\pi^2} \right] \left( \frac{32}{\pi^3} \frac{3}{\pi} - \frac{16.3}{\pi^4} \right) = -0.4928 \]

\[ \frac{16.3}{\pi^4} = 0.4928 \]

\[ vs. \quad 0.5 \]

\[ I_{\text{min (Trial)}} \quad I_{\text{min (Exact)}} \]
CHAPTER 7
More Practical Example

- ψ_{ij} defined on 2D grid

\[ \frac{1}{2} \int d^2x \left| \nabla \psi \right|^2 \Rightarrow \frac{1}{2} \sum_{\text{Squares } (i,j)} \Delta x \Delta y \left\{ \frac{1}{2\Delta x} \left[ \left( \psi_{i+1,j+1} + \psi_{i,j} \right) - \left( \psi_{i+1,j} + \psi_{i,j+1} \right) \right]^2 \right. \\
+ \left[ \frac{1}{2\Delta y} \left( \psi_{i+1,j+1} + \psi_{i,j+1} \right) - \left( \psi_{i+1,j} + \psi_{i,j+1} \right) \right]^2 \right\} \]

\[ \int d^3x \frac{\rho}{\epsilon_0} \psi = \sum_{\text{Square}} \Delta x \Delta y \frac{\rho_{ij} \psi_{ij}}{\epsilon_0} \]

\[ I \Rightarrow I \{ \psi_{ij} \} \text{ Minimize numerically converges to solution} \]