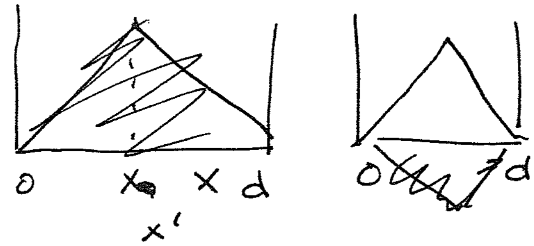
CHAPTER II

$$\frac{d^2 \overset{G(x,x_0)}{\cancel{\phi}}}{dx^2} = -4\pi\delta(x-x_0)$$



$$G_D(x,x') = \begin{cases} -4\pi\left(1-\frac{x}{d}\right)x' & 0<x'<x \\ -4\pi\left(1-\frac{x'}{d}\right)x & x<x'<d \end{cases}$$

Alternate representation in orthogonal function

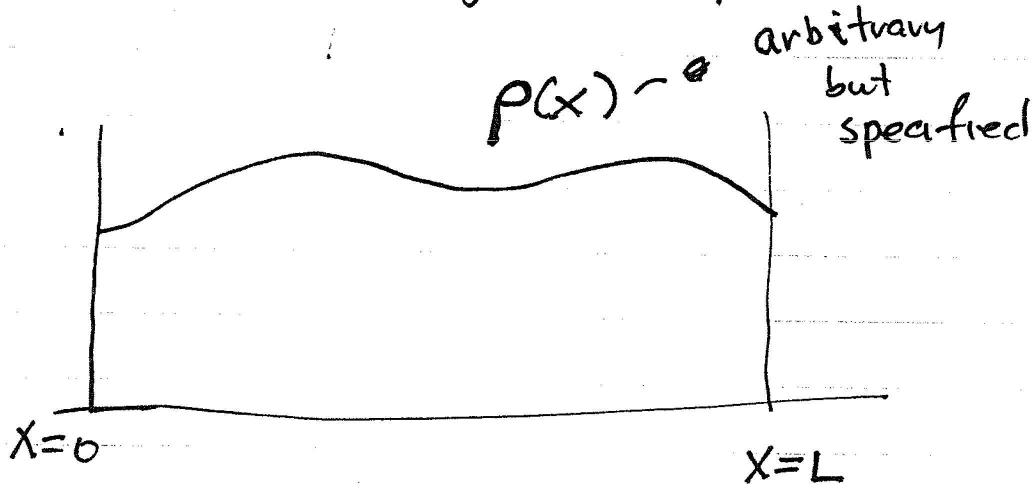
$$\phi(x) = \int_0^d dx'\, G_D(x,x')\frac{\rho(x')}{4\pi}$$

ORTHOGONAL FUNCTIONS

Solution of Poisson's equation based on expansion in terms of orthogonal functions

$$\nabla^2 \phi = -\cancel{4\pi}\rho(\underset{\sim}{x})/\epsilon_0$$

Consider the following 1D example



$$\frac{d^2\phi}{dx^2} = -4\pi\rho(x)$$

$$\phi(x=0) = 0$$

$$\phi(x=L) = 0$$

How to solve

what you would really ~~do~~ do is integrate (ordinary equation)

$$\frac{d\phi}{dx} = \left.\frac{d\phi}{dx}\right|_0 - 4\pi \int_0^x \rho(x')\,dx'$$

$$\phi(x) = \cancelto{0}{\phi(0)} + \int_0^x dx' \frac{d\phi}{dx'}$$

$$= \int_0^x dx' \left[\left.\frac{d\phi}{dx}\right|_0 - \cancel{4\pi} \int_0^{x'} dx'' \frac{\rho(x'')}{\epsilon_0} \right]$$

$$= x \left.\frac{d\phi}{dx}\right|_0 - \cancel{4\pi} \underbrace{\int_0^x dx' \int_0^{x'} dx'' \rho(x'') / \epsilon_0}_{\text{can be simplified}}$$

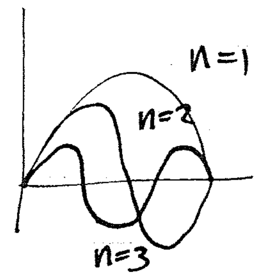
$$= x \left.\frac{d\phi}{dx}\right|_0 - \cancel{4\pi} \int_0^x dx'' \rho(x'')(x - x'') / \epsilon_0$$

pick $\left.\frac{d\phi}{dx}\right|_0 = 0$

Alternate method based on Fourier Series

write a Fourier sine series for $\phi(x)$

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$



choice of basis functions motivated by B.C's

all terms satisfy b.c.

substitute for $\phi(x)$ in P.E.
multiply by $\sin \frac{m\pi x}{L}$ and integrate from 0 to L

$$\int_0^L dx \, \sin \frac{m\pi x}{L} \left[+ \frac{d^2}{dx^2} \sum_n a_n \sin \frac{n\pi x}{L} \right]$$

assume can be differentiated term by term

$$\sum_n \left(\frac{n\pi}{L}\right)^2 a_n \overbrace{\int_0^L dx \, \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}}^{\begin{cases} \frac{L}{2} & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}}$$

$$= -4\pi L \int_0^L \frac{dx}{L} \sin\frac{m\pi x}{L} \rho(x)$$

algebraic equation $\qquad \frac{d^2}{dx^2} \sin\frac{m\pi x}{L} = -\left(\frac{m\pi}{L}\right)^2 \sin\frac{m\pi x}{L}$

$$\frac{L}{2}\left(\frac{m\pi}{L}\right)^2 a_m = -4\pi \cancel{\int_0^L} \frac{L}{2} \rho_m$$

also ~~or~~ orthogonal fn is eigen of Laplacian

where
$$\rho_m = \frac{2}{L}\int_0^L dx \, \sin\frac{m\pi x}{L} \rho(x)$$

Solution

$$\left.\begin{aligned} \rho(x) &= \sum_m \rho_m \sin\frac{m\pi x}{L} \\ \phi(x) &= \sum_m a_m \sin\frac{m\pi x}{L} \end{aligned}\right\}$$

both ϕ and ρ expanded in orthogonal functions

$$a_m = \frac{-4\pi\rho_m}{(m\pi/L)^2}$$

Orthogonal functions

$$f(x) = \sum_{n=0}^{\infty} a_n U_n(x)$$

$$a_n = \int_a^b dx'\, U_n(x') f(x')$$

$$f(x) = \sum_{n=0}^{\infty} \int_a^b dx'\, U_n(x') U_n(x) f(x')$$

complete set $\Rightarrow$ $\sum_{n=0}^{\infty} U_n(x') U_n(x) = \delta(x-x')$

Choice of basis functions

Choose a set which simplifies the differential equation

if solving Laplace Eqn $\nabla^2 \phi = 0$
choose $\nabla^2 U_n = 0$ [U_n not complete]

if solving Poisson Equation

$$\nabla^2 \phi = -\cancel{4\pi}\rho/\epsilon_0$$

choose ~~[illegible]~~ $\nabla^2 \cancel{\phi} u_n + k_n^2 u_n = 0$

* ~~manipulate pro~~

eigen value

* choose a set which satisfies appropriate Boundary conditions

~~manipulate pro~~

---

* exploit symmetry to simplify problem

* ~~practical~~ manipulate problem so that solution converges with minimum number of terms

Return to our example

$$\frac{d^2}{dx^2}\phi = -\cancel{4\pi}\rho(x)\cancel{/\epsilon_0} \qquad 0 < x < L$$

$$\phi(0) = \phi_0 \qquad \phi(L) = \phi_1$$

note

No brainer approach

expand $\phi(x)$ in basis $u_n(x)$
satisfy

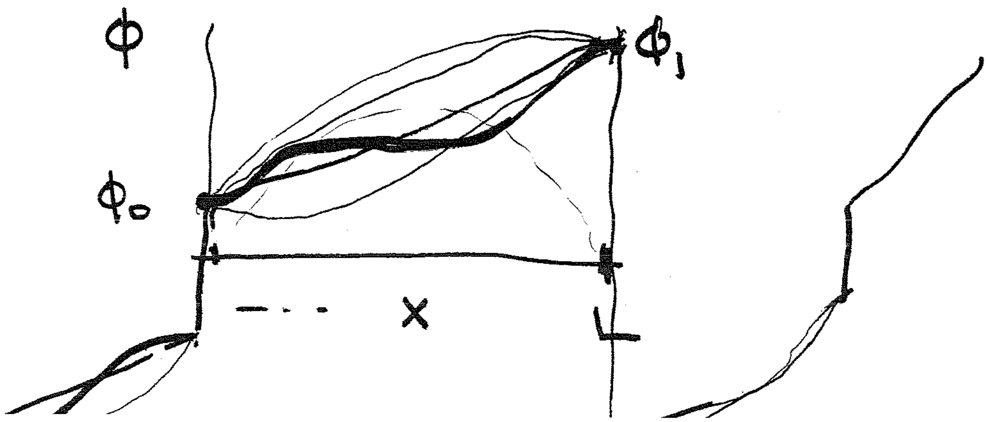
$$\frac{d^2 u_n}{dx^2} = -k_n^2 u_n \qquad \text{and} \qquad \boxed{\begin{array}{c} u_n(0) = 0 \\ u_n(L) = 0 \end{array}}$$

normalisation con

$$u_n(x) = C_n \sin \frac{n\pi x}{L} \qquad n = 1, 2, 3, \ldots$$

$$k_n = \frac{n\pi}{L} \qquad C_n = \sqrt{\frac{2}{L}} \qquad \int_0^L dx\, u_n^2(x) = 1$$

picture of
$\phi(x)$ we are
creating



~~substit~~ not absolutely convergent !

multiply equation by $u_n(x)$ &

integrate over x,

$$\int_0^L dx\, u_n(x) \frac{d^2\phi}{dx^2} = - \cancel{4\pi} \underbrace{\int_0^L dx\, u_n(x) \rho(x)}_{\rho_n} / \epsilon_0$$

$$\parallel$$

$$\underbrace{u_n(x) \frac{d\phi}{dx} \Bigg|_{0+\epsilon}^{L-\epsilon}}_{0} - \phi \frac{du_n}{dx} \Bigg|_{0+\epsilon}^{L-\epsilon} + \int_{0+\epsilon}^{L-\epsilon} dx\, \phi(x) \frac{d^2 u_n}{dx^2}$$

$$= - \cancel{4\pi} \rho_n / \epsilon_0$$

$u_n(x) \to 0$

$$\frac{du_n}{dx} = k_n C_n \cos k_n x$$

$$- k_n C_n \left[\phi_1 \cos n\pi - \phi_0 \right] + \int_{0+\epsilon}^{L-\epsilon} dx\, \phi(x) (-k_n^2 u_n)$$

$$= - \cancel{4\pi} \rho_n / \epsilon_0$$

$$\phi_n = \int dx\, u_n \phi(x)$$

$$+k_n^2\phi_n = +\cancel{4\pi}\rho_n/\epsilon_0 + k_n c_n\left[\phi_1 \cos n\pi - \phi_0\right]$$

$$\phi_n = \frac{\cancel{4\pi}\rho_n}{k_n^2\epsilon_0} + \frac{\overbrace{c_n}^{\sqrt{\frac{2}{L}}}}{k_n}\left[\phi_1 \cos n\pi - \phi_0\right]$$

$\sim \frac{1}{n^2}$

$\sim 1/n$ note poor convergence

$$k_n = \frac{n\pi}{L}$$

<u>Better approach</u>

$$\phi(x) = \phi_p(x) + \phi_h(x)$$

$$\frac{d^2\phi_h}{dx^2} = 0 \qquad \phi_h(0) = \phi_0$$

$$\phi_h(L) = \phi_1$$

$$\frac{d^2\phi_p}{dx^2} = -4\pi\rho(x) \qquad \phi_p(0) = \phi_p(L) = 0$$

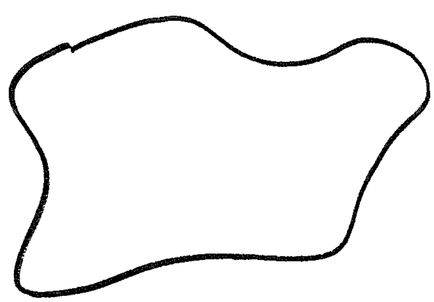
$$\phi_h(x) = \phi_0 + \frac{x}{L}(\phi_1 - \phi_0)$$

$$\boxed{\phi_p(x) = \sum_n \frac{4\pi\rho_n}{k_n^2} u_n(x)}$$

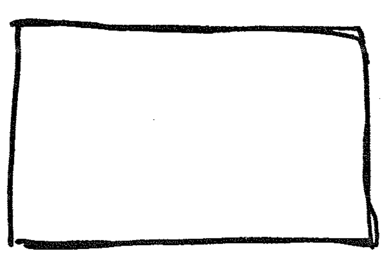
Solutions in higher dimensions

$$\nabla^2 u_n = -k_n^2 u_n$$

<u>how to find</u>



for general boundaries
~~not~~ no analytic solutions



for ~~symmetr~~ ~~separable~~ symmetric systems
separation of variables

CHAPTER 12

[C]omments on separation of variables

in 3

Basic Functions For Laplace's Eq.: Rectangular Coordinates

$$\nabla^2 h = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) h = 0$$

separation of variables: assume

$$h(x,y,z) = X(x)\,Y(y)\,Z(z)$$

go through avg

$$\underbrace{\frac{1}{X}\frac{\partial^2}{\partial x^2}X}_{\text{depends only on } x} + \underbrace{\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y}_{\text{only } y} + \underbrace{\frac{1}{Z}\frac{\partial^2}{\partial z^2}Z}_{\text{only } z} = 0$$

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X = -\alpha^2$$

$$\frac{\partial^2}{\partial x^2}X + \alpha^2 X = 0 \qquad X = e^{\pm i\alpha x} \qquad X = \cos\alpha x, \sin\alpha x$$

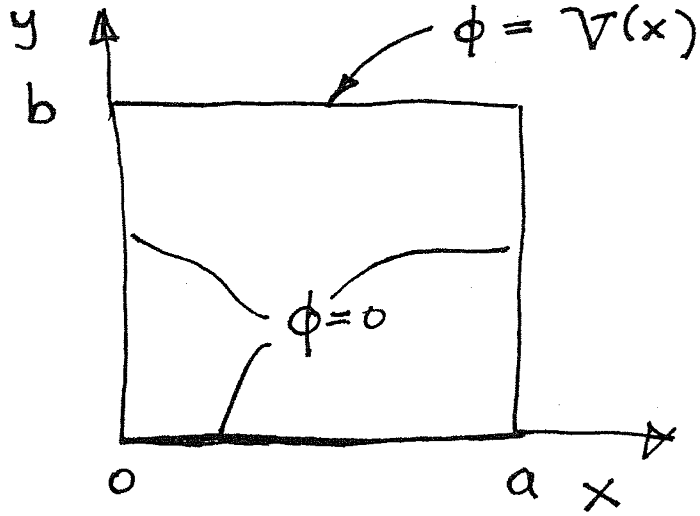
$$\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y = -\beta^2 \qquad Y = e^{\pm i\beta y}$$

$$\frac{\partial^2}{\partial z^2}Z - (\alpha^2+\beta^2)Z = 0 \qquad Z = e^{\pm\sqrt{\alpha^2+\beta^2}\,z}$$

$$\boxed{h(x,y,z) = e^{\pm i\alpha x}\, e^{\pm i\beta y}\, e^{\pm\sqrt{\alpha^2+\beta^2}\,z}}$$

- α and β are arbitrary functions (could be complex)
- note that can permute the variables x, y, z

Example Consider a two dimensional Domain



$$\nabla^2 \phi = 0$$

$$\phi(x,y) = \sum_n A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

why sin? satisfies bc at $x=0$ & $x=a$

why sinh? satisfies bc at $y=0$

(otherwise) $\sum_n \sin\left(\frac{n\pi x}{a}\right)\left[A_n \sinh\left(\frac{n\pi y}{a}\right) + B_n \cosh\left(\frac{n\pi y}{a}\right)\right]$

or $$\sum_n \sin\left(\frac{n\pi x}{a}\right)\left[C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}\right]$$

at $y = b$

$$\phi(x,b) = V(x) = \sum_n \sin\left(\frac{n\pi x}{a}\right) A_n \sinh\left(\frac{n\pi b}{a}\right)$$

$$V(x) = \sum_n V_n \sin\left(\frac{n\pi x}{a}\right)$$ FOURIER

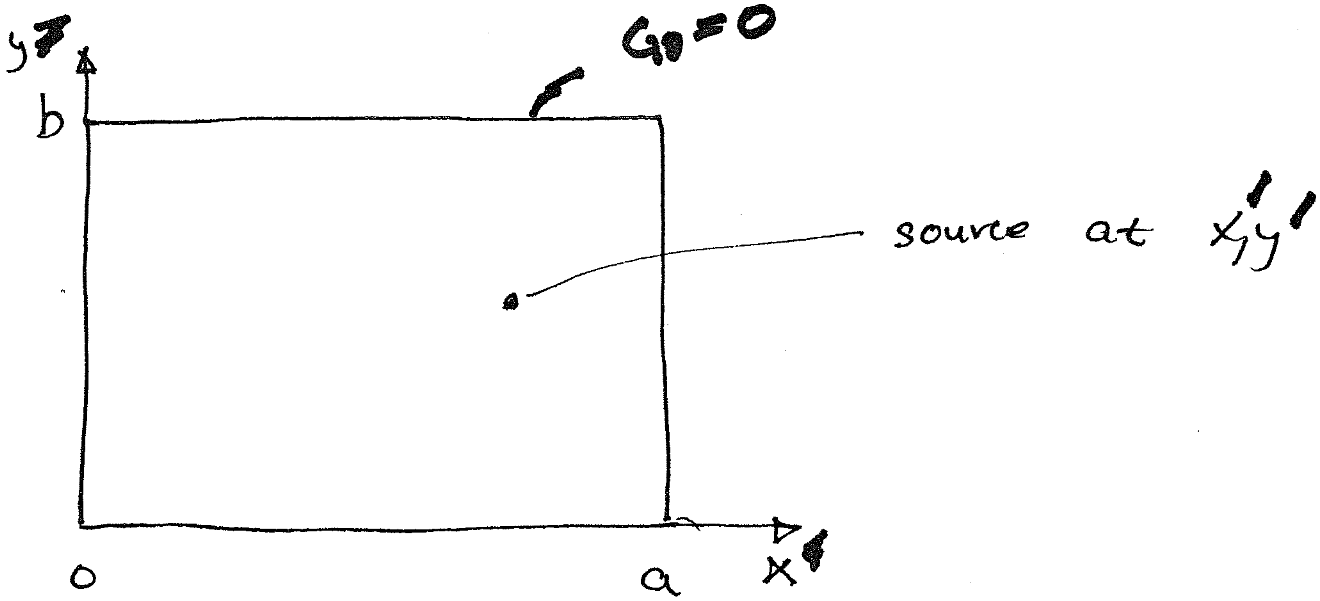
$$V_n = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right) V(x)\, dx$$

THUS $A_n \sinh\left(\frac{n\pi b}{a}\right) = V_n$

determines A_n

Dirichlet

Green's Function for a Rectangular box



$$\nabla^2 G(\underset{\sim}{x}, \underset{\sim}{x}') = -4\pi\,\delta(x-x')\,\delta(y-y')$$

Here we are not solving Laplacian so separation constants not equal

~~prob~~ ~~basis~~ $\nabla^2 u + k^2 u = 0$

$$G(\underset{\sim}{x}, \underset{\sim}{x}') = \sum_{m,n} G_{m,n}(x', y') \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}$$

$$\nabla'^2 G(x,x') = \cancel{-\sum_{m,n} (k_n^2 + k_m^2)}$$

$$= -\sum_{m,n} \left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \right) G_{m,n} \sin\frac{n\pi x'}{a} \sin\frac{m\pi y'}{b}$$

$$= -4\pi\, \delta(x-x')\delta(y-y')$$

multiply by $\sin\frac{p\pi x'}{a} \sin\frac{\ell\pi y'}{b}$

and integrate over x' & y'

only $n=p$, $m=\ell$ term survives

$$\int_0^a dx' \int_0^b dy' \sin^2\frac{p\pi x'}{a} \sin^2\frac{\ell\pi}{b} y' = \frac{ab}{4}$$

$$\int_0^a dx' \int_0^b dy' \sin\frac{p\pi x'}{a} \sin\frac{\ell\pi}{b} y' \,\delta(x-x')\delta(y-y')$$

$$= \sin\frac{p\pi x}{a} \sin\frac{\ell\pi y}{b}$$

THUS

$$-\frac{ab}{4}\left[\left(\frac{p\pi}{a}\right)^2+\left(\frac{l\pi}{b}\right)^2\right]G_{p,l}(x',y')$$

$$= -4\pi \sin\frac{p\pi x'}{a}\sin\frac{l\pi y'}{b}$$

determines $G_{p,l}$

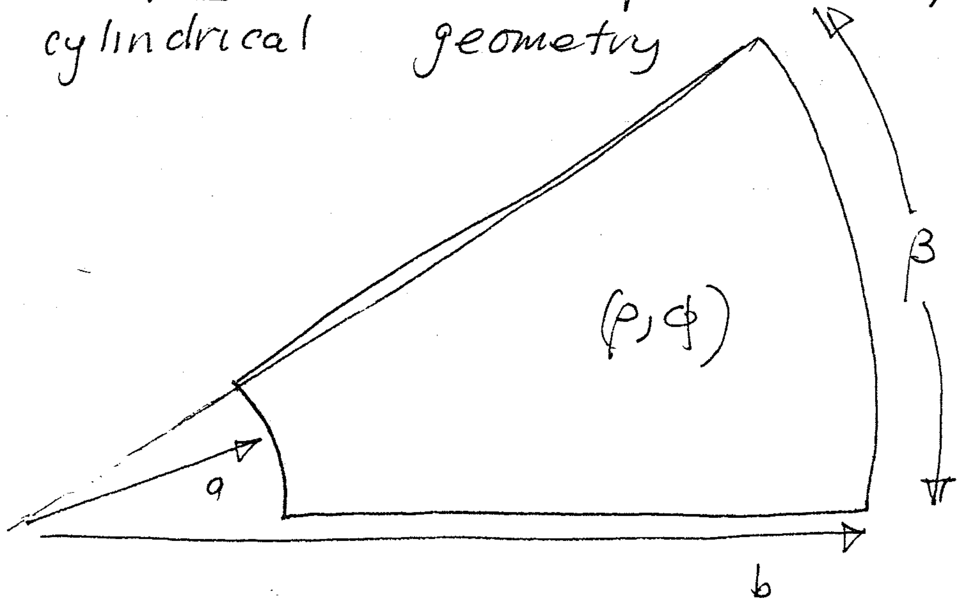
$$G(x,y,x',y') = \sum_{nm} \frac{16\pi}{ab}\frac{1}{\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]}$$

$$\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}$$

CHAPTER 13

Separation in Cylindrical Coord

Examples of Laplace's Equation in cylindrical geometry



Separation $\qquad \Phi(\rho, \varphi) = R(\rho)\psi(\varphi)$

$$\nabla^2 \Phi = 0 \qquad \text{Laplaces Equation}$$

$$\nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}$$

$$= \frac{R\psi}{\rho^2}\left\{ \underbrace{\frac{\rho}{R}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R}_{\text{separation constant } \nu^2} + \underbrace{\frac{1}{\psi}\frac{\partial^2}{\partial\varphi^2}\psi}_{-\nu^2} \right\}$$

ν^2 is arbitrary

for a particular $\nu^2 > 0$

~~possi~~ solutions can be written

$$\psi = C\exp(i\nu\varphi) \text{ , or } A\sin\nu\phi + B\cos\nu\varphi$$

$$R = \cancel{Ar^{\pm\nu}} \quad a\rho^{\nu} + b\rho^{-\nu}$$

arbitrary constants

~~general so~~

IF $\nu^2 = 0$

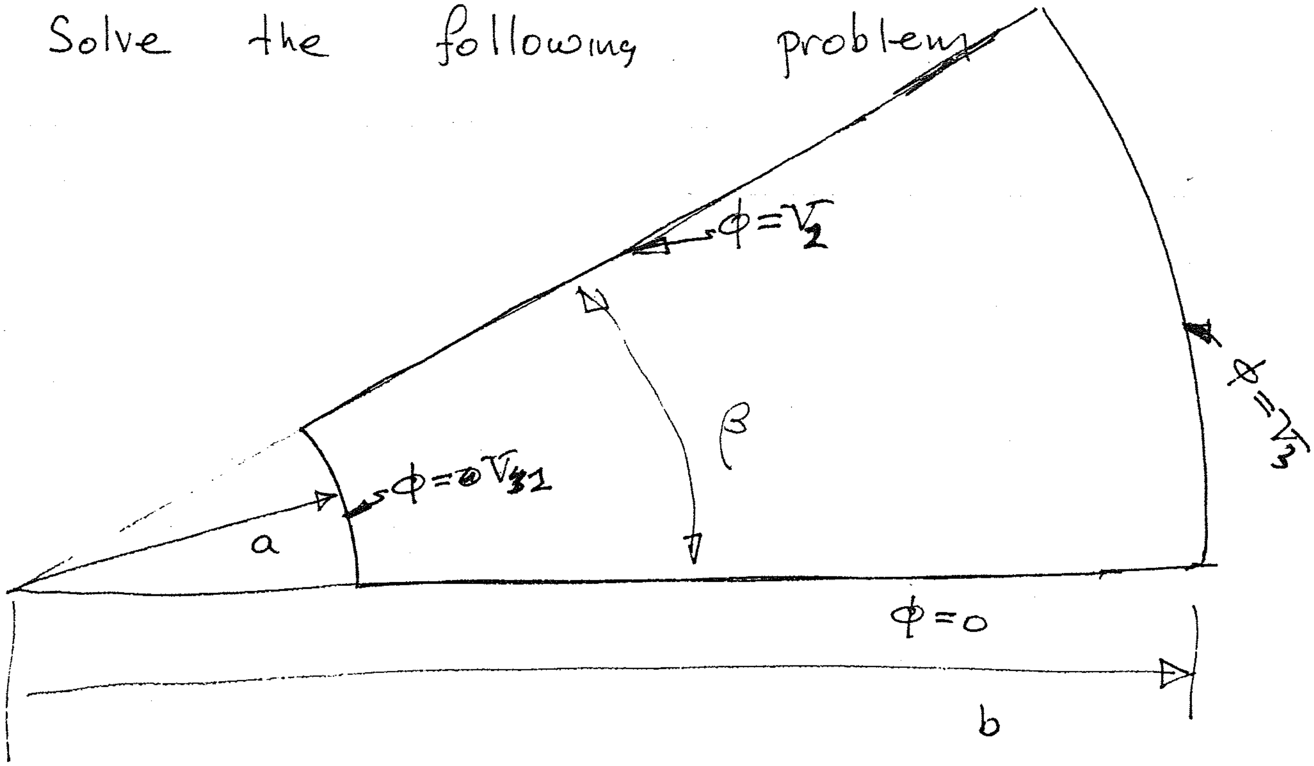
$$\frac{\partial^2\psi}{\partial\varphi^2} = 0 \qquad \psi = A_0 + B_0\varphi$$

$$\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}R = 0 \qquad R = a_0 + b_0\ln\rho$$

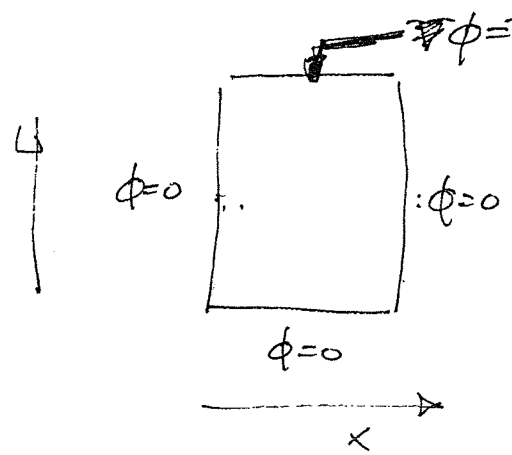
if Region of solution corresponds to full range of φ such that

$\psi(\phi + 2\pi) = \psi(\phi)$ (periodic) ν = integer

Solve the following problem



Similar to our problem



$$\phi = \sum_n a_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

$\cancel{\phi = B_0 r}$

start with a ~~separa~~ solution corresponding to zero separation constant,

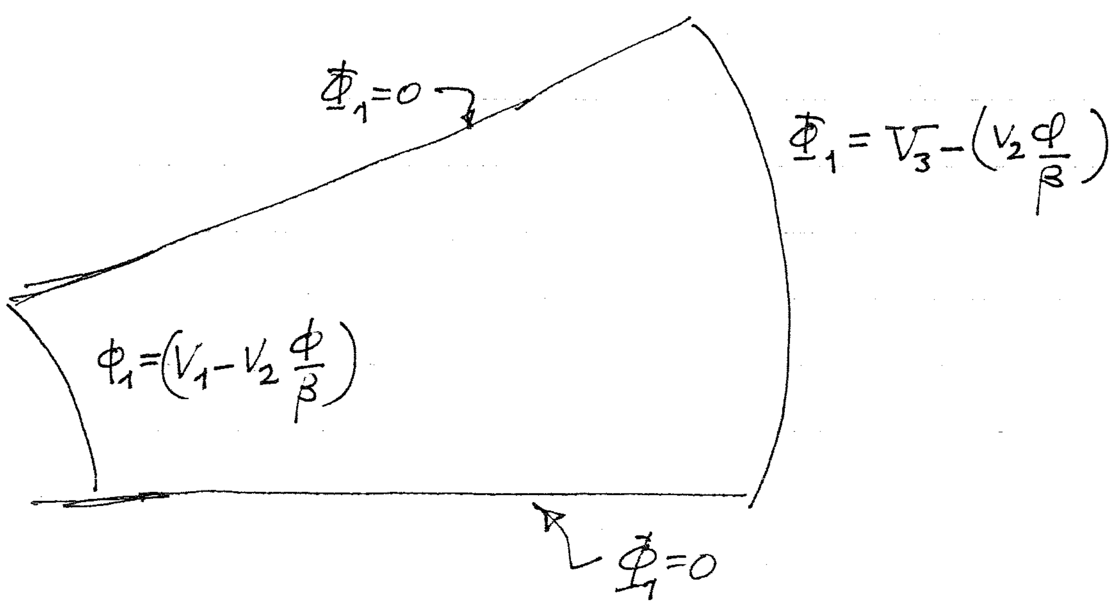
V13 $\qquad \psi = B_0 \varphi \qquad R = a_0$

why? this gives Φ = constant on upper boundary

$$\Phi = V_2 \frac{\varphi}{\beta} + \Phi_1$$

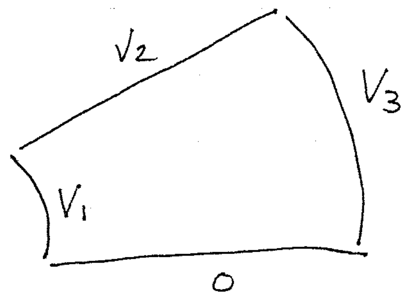
~~whe~~ $$\nabla^2 \Phi = \cancelto{0}{\nabla^2 V_2 \frac{\varphi}{\beta}} + \nabla^2 \Phi_1 = 0$$

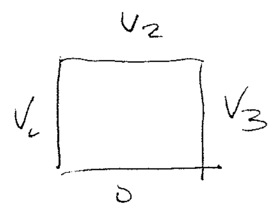
$$\nabla^2 \Phi_1 = 0$$



We are going to do the following problems:

1)





2) $$\nabla^2 G = -4\pi\,\delta(\underline{x}-\underline{x}') \quad \text{in 2D}$$

$$\Phi_1 = \sum_{n=1}^{\infty} \left[a_n \rho^{\nu_n} + b_n \rho^{-\nu_n} \right] \sin \nu_n \varphi$$

$$\nu_n = \frac{n\pi}{\beta}$$ n^{th} separation constant

at $\rho = a$

$$\Phi_1 = \left(V_1 - V_2 \frac{\varphi}{\beta}\right) = \sum_{n=1} \left[a_n a^{\nu_n} + b_n a^{-\nu_n} \right] \sin \nu_n \phi$$

multipy by $\sin \nu_m \varphi$ and integate o to β

$$\left[a_m a^{\nu_m} + b_m a^{-\nu_m} \right] \beta/2 = \int_0^\beta d\varphi \sin \nu_m \varphi \left[V_1 - V_2 \frac{\varphi}{\beta} \right]$$

$$\equiv \beta/2 \; d_m$$

$$\equiv \frac{\beta}{2} e_m$$

at $\rho = b$

$$\left[a_m b^{\nu_m} + b_m b^{-\nu_m} \right] \beta/2 = \int^\beta d\phi \sin \nu_m \phi \left[V_1 - V \frac{\varphi}{\beta} \right]$$

WRITE

$$\phi_1 = \sum_{n=1}^{\infty} \left[c_n \underbrace{\left[\left(\frac{\rho}{a}\right)^{\nu_n} - \left(\frac{a}{\rho}\right)^{\nu_n}\right]}_{=0 \text{ on } a} + d_n \underbrace{\left[\left(\frac{b}{\rho}\right)^{\nu_n} - \left(\frac{\rho}{b}\right)^{\nu_n}\right]}_{=0 \text{ on } b} \right]$$

$$\sin \nu_n \varphi$$

at $\rho = a$ $\quad \Phi_1 = V_1 - V_2 \frac{\varphi}{\beta}$

$$= \sum_{n=1}^{\infty} d_n \left[\left(\frac{b}{a}\right)^{\nu_n} - \left(\frac{a}{b}\right)^{\nu_n}\right] \sin \nu_n \varphi$$

multiply by $\sin \nu_m \varphi$ and integrate 0 to β

$$d_m \left[\left(\frac{b}{a}\right)^{\nu_m} - \left(\frac{a}{b}\right)^{\nu_m}\right] \frac{\beta}{2} = \int_0^\beta d\beta \sin \nu_m \varphi \left[V_1 - V_2 \frac{\varphi}{\beta}\right]$$

$$\equiv \frac{\beta}{2} \bar{d}_m \qquad \text{can be done}$$

Like wize

$$C_m \left[\left(\frac{b}{a}\right)^{\nu_m} - \left(\frac{a}{b}\right)^{\nu_m} \right] \frac{\beta}{2} = \int_0^{\beta} d\beta \, \sin \nu_m \varphi \left[V_3 - V_2 \frac{\varphi}{\beta} \right]$$

$$\equiv \frac{\beta}{2} e_m$$

lets suppose $b >\!\!>\!\!> a$

$$\phi_1 = \sum_{n=1}^{\infty} e_m$$

suppose $\quad b >> a, \rho$

and $\quad$ ~~$b \gg \rho \simeq a$~~

$$C_m \to 0 \quad \text{as} \quad \left(\frac{a}{b}\right)^{\nu_m}$$

$$\phi_1 \simeq \sum_{n=1}^{\infty} \left(\frac{a}{\rho}\right)^{\nu_m} d_n \sin \nu_n \varphi$$

when $\quad a = \rho \quad$ gives $\quad V_1 - V_2 \frac{\varphi}{\beta} \quad$ as it should

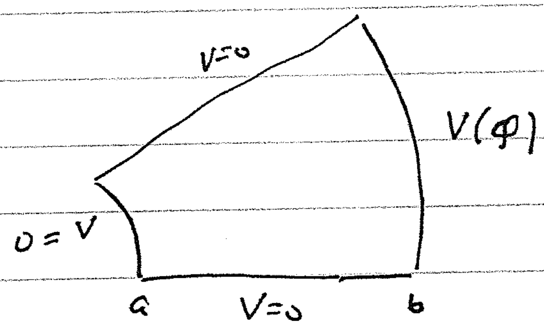
as $\quad \rho / a \to \infty \quad$ (but $\rho/b << 1$)

$\phi_1 \to 0$

as $b/a \to \infty$

$$d_m \sim \left(\frac{a}{b}\right)^{\nu_m}$$

$$c_m \sim \left(\frac{a}{b}\right)^{\nu_m}$$



$$\phi_{\cancel{2}} = \sum_{n=1}^{\infty} \left[c_n \left(\frac{\rho}{a}\right)^{\nu_n} + d_n \left(\frac{a}{\rho}\right)^{\nu_n} \right] \sin \nu_n \varphi$$

when ~~@~~ $\rho = a$ $\quad \phi_1 = 0 \qquad d_n = -c_n$

as $\rho = b$

$$V(\varphi) = \sum_n c_n \left[\left(\frac{b}{a}\right)^{\nu_n} - \left(\frac{a}{b}\right)^{\nu_n} \right] \sin \nu_n \phi$$

$$= \sum_n V_n \sin \nu_n \varphi$$

$$C_n = \frac{V_n}{\left[\left(\frac{b}{a}\right)^{\nu_n} - \left(\frac{a}{b}\right)^{\nu_n}\right]}$$

$$\phi = \sum_{n=1}^{\infty} \frac{\left[(\rho/a)^{\nu_n} - (a/\rho)^{\nu_n}\right]}{\left[\left(\frac{b}{a}\right)^{\nu_n} - \left(\frac{a}{b}\right)^{\nu_n}\right]} V_n \sin n\varphi$$

$$\frac{\left[(\rho/a)^{\nu_n} - (a/\rho)^{\nu_n}\right] V_n}{(b/a)^{\nu_n}}$$

$\nu_n = \frac{n\pi}{\beta}$ as $(b/a) \to \infty$

leading term is

one with smallest ν_n $(n=1)$

$$\boxed{\phi \simeq \left(\frac{(\rho/a) - (a/\rho)}{(b/a)}\right) V_1 \sin(\pi\phi/\beta)}$$

Green's Function in cylindrical coordinates (2Dimensional case)

$$\nabla^2 G = -4\pi\,\delta(\underset{\sim}{r}-\underset{\sim}{r}')$$

in two demensions ρ, ϕ

two dimension delta fun

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}G + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}G = -4\pi \boxed{\frac{\delta(\varphi-\varphi')\,\delta(\rho-\rho')}{\rho}}$$

$$\int d\phi' \rho' d\rho' \, \frac{\delta(\varphi-\phi')\,\delta(\rho-\rho')}{\rho'} = 1$$

assume periodic in φ

write $$G = \sum_{\nu=-\infty}^{\infty} \hat{G}_\nu(\rho,\rho',\varphi')\, e^{i\nu\phi}$$

$$e^{-i\nu\phi} \int \frac{d\phi}{2\pi}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\hat{G}_\nu - \frac{\nu^2}{\rho^2}\hat{G}_\nu = -4\pi\int_0^{2\pi}\frac{d\phi}{2\pi}\,\frac{\delta(\phi-\varphi')\,\delta(\rho-\rho')}{\rho}\,e^{-i\nu\phi}$$

multiply be $e^{i\nu\varphi}$

can do ϕ integral

$$= -4\pi \frac{1}{2\pi} e^{-i\nu\phi'} \frac{\delta(\rho-\rho')}{\rho}$$

let $\hat{G}_\nu = e^{-i\nu\phi'} G_\nu(\rho,\rho')$

$$G = \sum_{\bar{\nu}=-\infty}^{\infty} G_{\bar{\nu}} e^{i\bar{\nu}(\phi-\phi')}$$

$$\boxed{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}G_\nu - \frac{\nu^2}{\rho^2}G_\nu = -2\frac{\delta(\rho-\rho')}{\rho}}$$

How to solve such an equation

Option #1 Represent $G_\nu(\rho)$ in a complete set of functions on the interval $\rho=0$ $\rho=\infty$

Range is infinite separation constant will be continuous

$$G_\nu(\rho) = \int dk\, A_\nu(k)\, G_\nu(\rho, k)$$

Bessel transform
(Like Fourier transform)

where

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}G_\nu(\rho,k) - \frac{\nu^2}{\rho^2}G_\nu(\rho,k) + k^2 G_\nu(\rho,k) = 0$$

Bessel's equation

$$G_\nu(\rho,k) = J_\nu(k\rho)$$

Option #2 since we have an ordinary equation solve it on either side of δ function & match for $\rho < \rho'$

$$G_\nu = a_\nu \rho^{|\nu|} + b_\nu \rho^{-|\nu|}$$

as $\rho \to 0$ G_ν finite requires

we set $b_\nu = 0$

$$G_\nu = a_\nu \rho^{|\nu|} \qquad \rho < \rho'$$

for $\rho > \rho'$

$$G_\nu = c_\nu \rho^{|\nu|} + d_\nu \rho^{-|\nu|}$$

as $\rho \to \infty$ G finite gives $c_\nu = 0$.

$$G_\nu = d_\nu \rho^{-|\nu|}$$

conditions at $\rho \simeq \rho'$

G is continuous

but $\frac{\partial G}{\partial \rho}$ is discontinuous

why

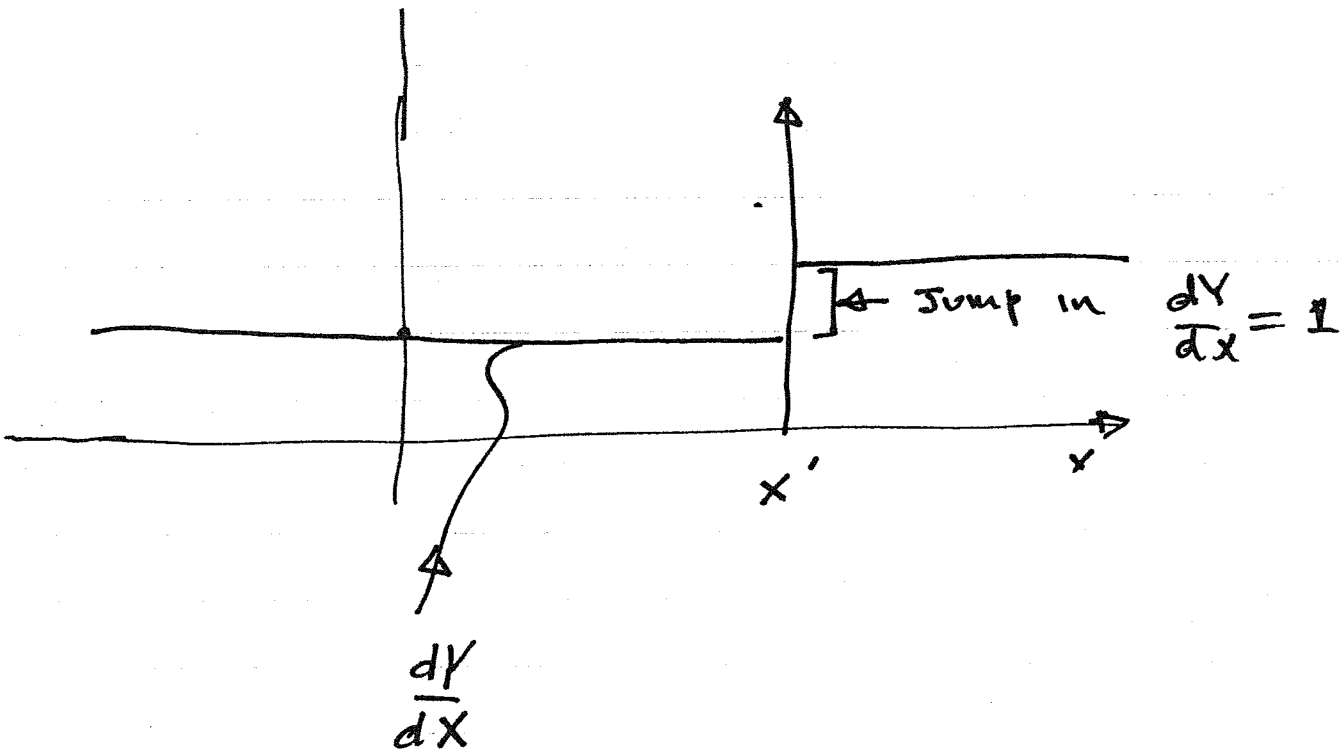
consider the equation

$$\frac{d^2}{dx^2} Y = \delta(x-x')$$

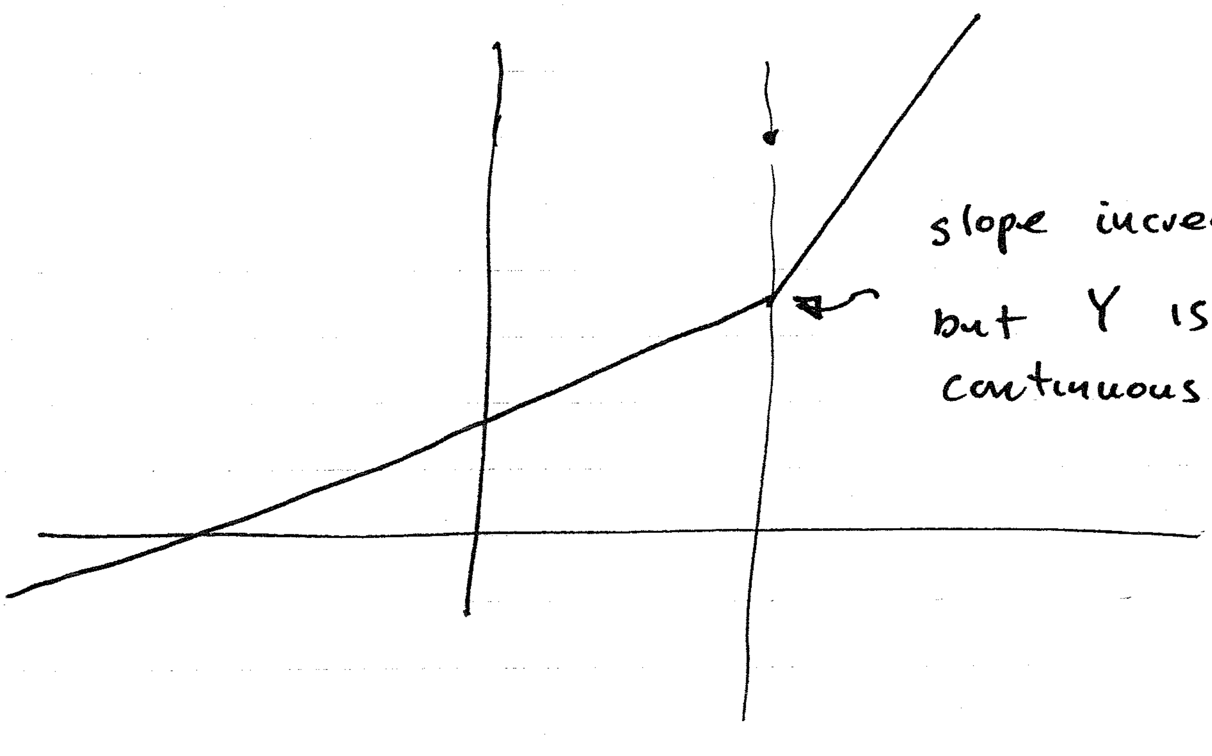
$$\frac{dY}{dx} = \int_{-\infty}^{x} dx' \, \delta(x-x')$$

$$= \left.\frac{dY}{dx}\right|_{-\infty} \qquad x < x'$$

$$= \left.\frac{dY}{dx}\right|_{-\infty} + 1 \qquad x > x'$$



integrate once more to get Y



slope increases but Y is continuous

continuity of G_ν

$$a_\nu \rho'^{|\nu|} = d_\nu \rho'^{-|\nu|}$$

discontinuity of $\partial G / \partial \rho$

for ρ near ρ'

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon} d\rho \left[\frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} G_\nu - \frac{\nu^2}{\rho} G_\nu \right] = -2 \int_{\rho'-\epsilon}^{\rho'+\epsilon} d\rho\, \delta(\rho - \rho')$$

$\epsilon \to 0$

$$\rho \frac{\partial}{\partial \rho} G_\nu \Big|_{\rho'-\epsilon}^{\rho+\epsilon} \quad \sim \epsilon \qquad = \quad -2$$

Jump

$$-|\nu| d_\nu \rho'^{-|\nu|} - |\nu| a_\nu \rho'^{|\nu|} = -2$$

$= d_\nu \rho'^{-|\nu|}$

$f \ldots$

$$d_\nu = \frac{1}{|\nu|} \{\rho'\}^{-|\nu|}$$

$$a_\nu = \frac{1}{|\nu|} \rho'^{-|\nu|}$$

$$G_\nu(\rho,\rho') = \frac{1}{|\nu|} \begin{cases} \left(\frac{\rho}{\rho'}\right)^{|\nu|} & \rho < \rho' \\ \left(\frac{\rho'}{\rho}\right)^{+|\nu|} & \rho > \rho' \end{cases}$$

$$G(\rho,\rho',\phi,\phi') = \sum_{\nu=-\infty}^{\infty} \frac{1}{|\nu|} e^{i\nu(\phi-\phi')} \begin{cases} \left(\frac{\rho}{\rho'}\right)^{|\nu|} & \rho<\rho' \\ \left(\frac{\rho'}{\rho}\right)^{|\nu|} & \rho>\rho' \end{cases}$$

symmetric $\rho \leftrightarrow \rho'$

~~what if $\nu = 0$ take limit~~

must do $\nu = 0$ dem

$G = a \qquad \rho < \rho'$

$G = b + c \ln \rho \qquad \rho > \rho'$ [diverges as $\rho \to \infty$]

Match G jump $\frac{\partial G}{\partial \rho}$

potential due to line charge diverges

$G = a \qquad \rho < \rho'$

$G = a + c \ln(\rho/\rho') \qquad \rho > \rho'$

constant undetermined

$\rho \frac{c}{\rho} = -2$

$c = -2$

$G = 0 \qquad \rho = b$

$G = 0 \qquad \rho = a$

$\rho < \rho'$

$G = \ln \frac{\rho}{a}$

$\rho > \rho' \quad G = \ln(\rho$ …

$\nu = 0$ <u>case</u>

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial G}{\partial\rho} = -2\,\frac{\delta(\rho-\rho')}{\rho}$$

integrate $$\int_0^{\rho} \rho' d\rho' \left\{ \frac{1}{\rho'}\frac{\partial}{\partial\rho'}\rho'\frac{\partial G}{\partial\rho'} \right\} = \begin{cases} -2 & \rho>\rho' \\ 0 & \end{cases}$$

$$\rho\frac{\partial G}{\partial\rho} - \cancel{0\left.\frac{\partial G}{\partial\rho}\right|_0}^{\,0} = \begin{cases} -2 & \rho>\rho' \\ 0 & \rho<\rho' \end{cases}$$

$$\frac{\partial G}{\partial\rho} = \begin{matrix} -2/\rho & \rho>\rho' \\ 0 & \rho<\rho' \end{matrix}$$

$$G(\rho) = G(0) \qquad \rho<\rho'$$

$$G(\rho) = G(0) - 2\ln\frac{\rho}{\rho'} \qquad \rho>\rho'$$

note $G \not\to 0$ as $\rho\to\infty$