* Using Coulomb's law to find the electric field due to a uniform sphere of charge

Coulomb's law

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{d^3r' \rho(r') (\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3}$$

Uniform sphere of charge

$$\rho(\underline{r}') = \begin{cases} \rho_0 & |r'| = r' < a \\ 0 & r' > a \end{cases}$$

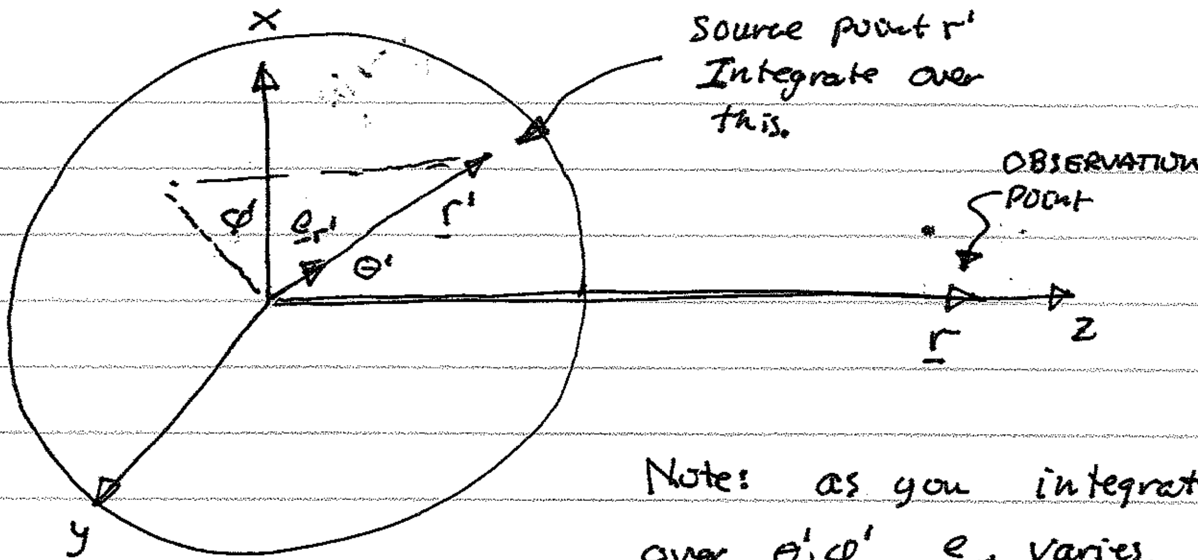
Locate observation point $\underline{r} = z\, \underset{\sim}{e}_z$ ← unit vector

Defines z-axis

Integrate over $\underline{r}'$ using spherical coordinates

$\underline{r}' = r' \underset{\sim}{e}_{r'}$ ← unit vector

$$d^3r' = r'^2 dr' \, d\varphi' \sin\theta' d\theta'$$



Note: as you integrate over θ', φ' $\underline{e}_{r'}$ varies, i.e. changes direction.

Find $E_z = \underline{e}_z \cdot \underline{E}$ (other components vanish due to symmetry)

$$E_z = \underline{e}_z \cdot \underline{E} = \frac{1}{4\pi\epsilon_0} \int_0^a r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\varphi'$$

$$\times \frac{\rho_0 \left(\underline{e}_z \cdot \underline{r} - \underline{e}_z \cdot \underline{r}' \right)}{|\underline{r} - \underline{r}'|^3}$$

EVALUATE QUANTITIES IN INTEGRAND

$\underline{e}_z \cdot \underline{r} = \underline{e}_z \cdot z \underline{e}_z = z$ $\qquad$ $\underline{e}_z \cdot \underline{r}' = \underline{e}_z \cdot \underline{e}_{r'} r' = \cos\theta' r'$

$|\underline{r} - \underline{r}'|^2 = (\underline{r} - \underline{r}') \cdot (\underline{r} - \underline{r}') = z^2 + r'^2 - 2\underline{r} \cdot \underline{r}' = z^2 + r'^2 - 2r'z\cos\theta'$

$$E_z = \frac{\rho_0}{4\pi\epsilon_0}\int_0^a r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\varphi' \frac{(z - r'\cos\theta')}{\left[z^2 + r'^2 - 2r'z\cos\theta'\right]^{3/2}}$$

1. Do φ' integral $\quad \int_0^{2\pi} d\varphi' \rightarrow 2\pi$

2. Do θ' integral

$$I = \int_0^\pi \sin\theta' d\theta' \frac{(z - r'\cos\theta')}{\left[z^2 + r'^2 - 2r'z\cos\theta'\right]^{3/2}} \overset{*}{=} \begin{cases} \frac{2}{z^2} & r' < z \\ 0 & r' > z \end{cases}$$

* See appendix

3.
$$E_z(z) = \frac{\rho_0 2\pi}{4\pi\epsilon_0}\int_0^a r'^2 dr' \begin{cases} \frac{2}{z^2} & r' < z \\ 0 & r' > z \end{cases}$$

IF $z > a$

$$E_z(z) = \frac{1}{4\pi\epsilon_0}\frac{4\pi\rho_0}{z^2}\int_0^a r'^2 dr' = \frac{Q}{4\pi\epsilon_0 z^2}$$

$$Q = \frac{4}{3}\pi a^3$$

If $z < a$

$$E_z(z) = \frac{\rho_0}{4\pi\epsilon_0} 4\pi \int_0^z \frac{r'^2 dr'}{z^2} = \frac{Q(z)}{4\pi\epsilon_0 z^2}$$

$Q(z) = \frac{4}{3}\pi z^3$ charge enclosed for $r' = z$

Appendix

$$I = \int_0^{\pi} \frac{\sin\theta' d\theta' \; (Z - r'\cos\theta')}{\left[Z^2 + r'^2 - 2r'Z\cos\theta'\right]^{3/2}}$$

let $\quad \omega(\theta') = Z^2 + r'^2 - 2r'Z\cos\theta'$

$$d\omega = 2r'Z \sin\theta' d\theta'$$

$$r'\cos\theta' = -\frac{\omega - (Z^2 + r'^2)}{2Z}$$

$$\omega(\theta'=0) = \omega_- = Z^2 + r'^2 - 2r'Z = (Z - r')^2$$

$$\omega(\theta'=\pi) = \omega_+ = Z^2 + r'^2 + 2r'Z = (Z + r')^2$$

$$I = \int_{\omega_-}^{\omega_+} \frac{d\omega}{2r'Z} \left(Z + \frac{\omega - (Z^2 + r'^2)}{2Z}\right) \frac{1}{\omega^{3/2}}$$

$$= \frac{1}{4r'Z^2} \int_{\omega_-}^{\omega_+} \frac{d\omega}{\omega^{3/2}} \left(\omega + Z^2 - r'^2\right)$$

$$= \frac{1}{4r'Z^2} \cancel{\int_{\omega_-}^{\omega_+} d\omega} \left[2\left(\omega^{1/2} - \frac{Z^2 r'^2}{\omega^{1/2}}\right)\Bigg|_{\omega_-}^{\omega_+} \right] =$$

$$\omega_\pm = r'^2 + z^2 \pm 2r'z$$

$$I = \frac{1}{2r'z^2}\left[\frac{|\omega_+| - (z^2 - r'^2)}{|\omega_+|^{1/2}} - \frac{|\omega_-| - (z^2 - r'^2)}{|\omega_-|^{1/2}}\right]$$

$$I = \frac{1}{2r'z^2}\left[\frac{2r'^2 + 2r'z}{|r'+z|} - \frac{2r'^2 - 2r'z}{|r'-z|}\right]$$

~~if z~~ r'

$$I = \cancel{2}\,\frac{1}{z^2}\left[\frac{r'+z}{|r'+z|} + \frac{z-r'}{|z-r'|}\right]$$

if $z > r'$ | if $z < r'$

$I = \frac{2}{z^2}$ | $I = 0$