The differential equation (8) is of second order and so must have a second solution with a second arbitrary constant. (The sine and cosine constitute the two solutions for the simple harmonic motion equation.) This solution cannot be obtained by the power series method outlined above, since a general study of differential equations would show that at least one of the two independent solutions of (8) must have a singularity at $r = 0$. There are several methods for obtaining this second solution, all too detailed to be included here, and several different forms for the solution. One form for the second solution (any of which may be called Bessel functions of second kind, order zero) easily found in tables is

$$N_0(v) = \frac{2}{\pi}\ln\left(\frac{\gamma v}{2}\right)J_0(v) - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m(v/2)^{2m}}{(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{m}\right). \quad (12)$$

The constant $\ln \gamma = 0.5772\ldots$ is Euler's constant. In general, then,

$$R = C_1J_0(Tr) + C_2N_0(Tr) \quad (13)$$

is the solution to (8), with

$$Z = C_3 \sinh (Tz) + C_4 \cosh (Tz) \quad (14)$$

as the corresponding form for the solution to (7). It should be noted from (12) that $N_0(Tr)$, the second solution to R, becomes infinite at $r = 0$, so it cannot be present in any problem for which $r = 0$ is included in the region over which the solution applies.

2. If T^2 is negative, let $T^2 = -\tau^2$ or $T = j\tau$, where τ is real, and (8) may be written

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \tau^2 R = 0. \quad (15)$$

The series (10) is still a solution, and T in (10) may be replaced by $j\tau$. Since all powers of the series are even, imaginaries disappear, and a new series is obtained which is real and also convergent. That is,

$$J_0(jv) = 1 + \left(\frac{v}{2}\right)^2 + \frac{(v/2)^4}{(2!)^2} + \frac{(v/2)^6}{(3!)^2} + \cdots. \quad (16)$$

Values of $J_0(jv)$ may be calculated for various values of v from such a series; these are also tabulated in the references. The defined function is denoted $I_0(v)$ in many of the references. Thus a solution to (15) is

$$R = C_1'J_0(j\tau r) \equiv C_1'I_0(\tau r). \quad (17)$$

There must also be a second solution in this case, and, since it is usually not taken simply as $N_0(j\tau r)$, the choice of this will be discussed in a later article (3.26). One of the forms for the second solution in this case is denoted $K_0(\tau r)$, so that the general solution to (15) may be written

$$R = C_1'I_0(\tau r) + C_2'K_0(\tau r). \quad (18)$$

The second solution K_0 becomes infinite at $r = 0$ just as does N_0, and so will not be required in the simple examples immediately following which include the axis $r = 0$ in the range over which the solution is to apply. The solution to the z equation (7) when $T^2 = -\tau^2$ is

$$Z = C_3' \sin \tau z + C_4' \cos \tau z. \quad (19)$$

Summarizing, either of the following forms satisfies Laplace's equation in the two cylindrical coordinates r and z:

$$\Phi(r, z) = [C_1J_0(Tr) + C_2N_0(Tr)][C_3 \sinh Tz + C_4 \cosh Tz] \quad (20)$$

$$\Phi(r, z) = [C_1'I_0(\tau r) + C_2'K_0(\tau r)][C_3' \sin \tau z + C_4' \cos \tau z]. \quad (21)$$

As was the case with the rectangular harmonics, the two forms are not really different since (20) includes (21) if T is allowed to become imaginary, but the two separate ways of writing the solution are useful, as will be demonstrated in following examples. The case with no assumed symmetries is discussed in Art. 3.26.

Problem

3.25 Demonstrate that the series (10) does satisfy the differential equation (8).

3.26 Bessel Functions

In Art. 3.25 an example of a Bessel Function was shown as a solution of the differential equation 3.25(8) which describes the radial variations in Laplace's equation for axially symmetric fields where a product solution is assumed. This is just one of a whole family of functions which are solutions of the general Bessel differential equation.

Bessel Functions with Real Arguments. For certain problems, as, for example, the solution for field between the two halves of a longitudinally split cylinder, it may be necessary to retain the ϕ variations in the equation. The solution may be assumed in product form again, RZF_ϕ, where R is a function of r alone, Z of z alone, and F_ϕ of ϕ alone. Z has solutions in exponentials or sinusoids as before, and F_ϕ may also be satisfied by sinusoids:

$$Z = Ce^{Tz} + De^{-Tz} \quad (1)$$

$$F_\phi = E \cos \nu\phi + F \sin \nu\phi. \quad (2)$$

The differential equation for R is then slightly different from the zero-order Bessel equation obtained previously:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(T^2 - \frac{\nu^2}{r^2}\right)R = 0. \quad (3$$

It is apparent at once that Eq. 3.25(8) is a special case of this more general equation, that is, $\nu = 0$. A series solution to the general equation carried through as in Art. 3.25 shows that the function defined by the series

$$J_\nu(Tr) = \sum_{m=0}^{\infty} \frac{(-1)^m (Tr/2)^{\nu+2m}}{m!\,\Gamma(\nu + m + 1)} \tag{4}$$

is a solution to the equation.

$\Gamma(\nu + m + 1)$ is the gamma function of $(\nu + m + 1)$ and, for ν integral, is equivalent to the factorial of $(\nu + m)$. Also for ν non-integral, values of this gamma function are tabulated. If ν is an integer n,

$$J_n(Tr) = \sum_{m=0}^{\infty} \frac{(-1)^m (Tr/2)^{n+2m}}{m!\,(n + m)!}. \tag{5}$$

A few of these functions are plotted in Fig. 3.26*a*. Similarly, a second independent solution[9] to the equation is

$$N_\nu(Tr) = \frac{\cos \nu\pi J_\nu(Tr) - J_{-\nu}(Tr)}{\sin \nu\pi}. \tag{6}$$

As may be noted in Fig. 3.26*b* these are infinite at the origin. So a complete solution to (3) may be written,

$$R = AJ_\nu(Tr) + BN_\nu(Tr). \tag{7}$$

The constant ν is known as the order of the equation. J_ν is then called a Bessel function of first kind, order ν; N_ν is a Bessel function of second kind, order ν. Of most interest for this chapter are cases in which $\nu = n$, an integer.

It is useful to keep in mind that, in the physical problem considered here, ν is the number of radians of the sinusoidal variation of the potential per radian of angle about the axis. For different applications of Bessel functions, ν has other significances.

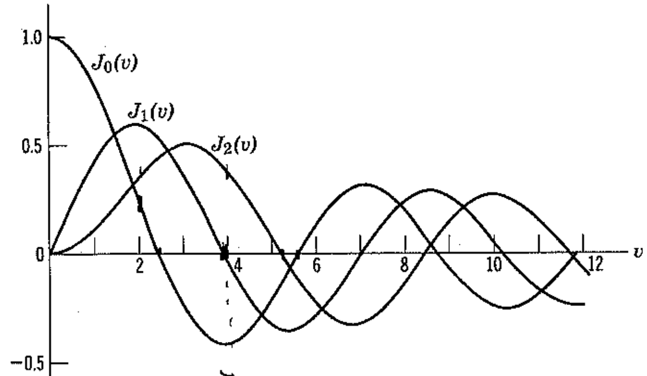
The functions $J_\nu(v)$ and $N_\nu(v)$ are tabulated in the references.[10,11,12] Some care should be observed in using these references, for there is a wide variation in notation for the second solution, and not all the functions used are equivalent, since they differ in the values of arbitrary constants selected for the series. The $N_\nu(v)$ is chosen here because it is the form most common in current mathematical physics, and also the form most commonly tabulated. It is equivalent to the $Y_\nu(v)$ used by Watson and by McLachlan. Of course, it is quite proper to use any one of the second solutions throughout a given problem, since all the differences will be absorbed in the arbitrary constants of the problem, and the same final numerical result will always be obtained; but is is necessary to be consistent in the use of only one of these throughout any given analysis.

[9] If ν is non-integral, $J_{-\nu}$ is not linearly related to J_ν, and it is then proper to use either $J_{-\nu}$ or N_ν as the second solution; for ν integral, N_ν must be used. Equation (6) is indeterminate for ν integral but is subject to evaluation by usual methods.

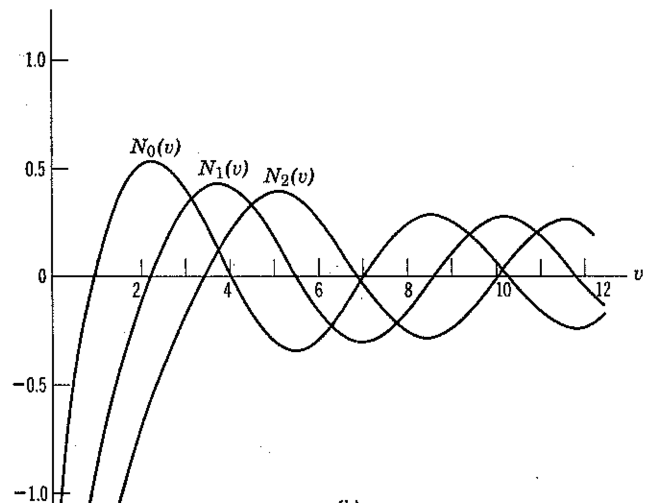
[10] E. Jahnke, F. Emde, and F. Lösch, *Tables of Higher Functions*, 6th ed. revised by F. Lösch, McGraw-Hill, New York, 1960.

[11] G. N. Watson, *Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Macmillan, New York, 1944.

[12] N. W. McLachlan, *Bessel Functions for Engineers*, 2nd ed., Oxford Clarendon Press, New York, 1955.



(*a*)



(*b*)

Fig. 3.26 (*a*) Bessel functions of the first kind. (*b*) Bessel functions of the second kind.

It is of interest to observe the similarity between (3) and the simple harmonic equation, the solutions of which are sinusoids. The difference between these two differential equations lies in the term $(1/r)(dR/dr)$ which produces its major effect as $r \to 0$. Note that for regions far removed from the axis as, for example, near the outer edge of Fig. 3.06*a*, the region bounded by surfaces of a cylindrical coordinate system approximates a cube. For these reasons, it may be expected that, away from the origin, the Bessel functions are similar to sinusoids. That this is true may be seen in Figs. 3.26*a* and *b*. For large values of the arguments, the Bessel functions approach sinusoids with magnitude decreasing as the square root of radius. For example,

$$J_\nu(Tr) \underset{Tr\to\infty}{=} \sqrt{\frac{2}{\pi Tr}} \cos\left(Tr - \frac{\pi}{4} - \frac{\nu\pi}{2}\right)$$

and the second kind, $N_\nu(Tr)$ approaches a sine variation with the same argument.

Hankel Functions. It is sometimes convenient to take solutions to the simple harmonic equation in the form of complex exponentials rather than sinusoids. That is, the solution of

$$\frac{d^2Z}{dz^2} + K^2Z = 0 \tag{8}$$

can be written as

$$Z = Ae^{+jKz} + Be^{-jKz} \tag{9}$$

where

$$e^{\pm jKz} = \cos Kz \pm j \sin Kz. \tag{10}$$

Since the complex exponentials are linear combinations of cosine and sine functions, we may also write the general solution of (8) as

$$Z = A'e^{jKz} + B' \sin Kz$$

or other combinations.

Similarly, it is convenient to define new Bessel functions which are linear combinations of the $J_\nu(Tr)$ and $N_\nu(Tr)$ functions. By direct analogy with the definition (10) of the complex exponential, we write

$$H_\nu^{(1)}(Tr) = J_\nu(Tr) + jN_\nu(Tr) \tag{11}$$

$$H_\nu^{(2)}(Tr) = J_\nu(Tr) - jN_\nu(Tr). \tag{12}$$

These are called *Hankel functions* of the first and second kinds, respectively. Since they both contain the function $N_\nu(Tr)$, they are both singular at $r = 0$. For large values of the argument, these can be approximated by complex exponentials with magnitude decreasing as square root of radius. For example,

$$H_\nu^{(1)}(Tr) \underset{Tr\to\infty}{=} \sqrt{2/(\pi Tr)}\, e^{j(Tr-\pi/4-\nu\pi/2)}.$$

This asymptotic form suggests that Hankel functions may be useful in wave propagation problems, as the complex exponential was in wave propagation on transmission lines in Chapter 1. We shall see more of these functions applied to wave propagation problems in later chapters. It is also sometimes convenient to use Hankel functions as alternate independent solutions in static problems. Complete solutions of (3) may be written in a variety of ways using combinations of Bessel and Hankel functions.

Bessel and Hankel Functions of Imaginary Arguments. If T is imaginary, $T = j\tau$, as in Eq. 3.25(15), (3) becomes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\tau^2 + \frac{\nu^2}{r^2}\right)R = 0. \tag{13}$$

The solution in (3) is valid here if T is replaced by $j\tau$ in the definitions of $J_\nu(Tr)$ and $N_\nu(Tr)$. In this case $N_\nu(j\tau r)$ is complex and so requires two numbers for each value of the argument whereas $j^{-\nu}J_\nu(j\tau r)$ is always a purely real number. It is convenient to replace $N_\nu(j\tau r)$ by a Hankel function. The quantity $j^{\nu+1}H_\nu^{(1)}(j\tau r)$ is also purely real and so requires tabulation of only one value for each value of the argument. If ν is not an integer, $j^\nu J_{-\nu}(j\tau r)$ is independent of $j^{-\nu}J_\nu(j\tau r)$ and may be used as a second solution. Thus, for nonintegral ν two possible complete solutions are

$$R = A_2J_\nu(j\tau r) + B_2J_{-\nu}(j\tau r) \tag{14}$$

and

$$R = A_3J_\nu(j\tau r) + B_3H_\nu^{(1)}(j\tau r) \tag{15}$$

where powers of j are included in the constants. For $\nu = n$, an integer, the two solutions in (14) are not independent but (15) is still a valid solution.

It is common practice to denote these solutions as

$$I_{\pm\nu}(v) = j^{\mp\nu}J_{\pm\nu}(jv) \tag{16}$$

$$K_\nu(v) = \frac{\pi}{2}\, j^{\nu+1}H_\nu^{(1)}(jv), \tag{17}$$

where $v = \tau r$.

As is noted in Art. 3.27 some of the formulas relating Bessel functions and Hankel functions must be changed for these *modified* Bessel functions. Special cases of these functions were seen as $I_0(\tau r)$ and $K_0(\tau r)$ in Art. 3.25 for the axially symmetric field. The forms of $I_\nu(\tau r)$ and $K_\nu(\tau r)$ for $\nu = 0, 1$ are shown in Fig. 3.26*c*. As is suggested by these curves, the asymptotic forms of the modified Bessel functions are related to growing and decaying real exponentials. For example,

$$K_\nu(\tau r) \underset{\tau r\to\infty}{\to} \sqrt{\frac{\pi}{2\tau r}}\, e^{-\tau r}.$$

It is also clear from the figure that $K_\nu(\tau r)$ is singular at the origin.

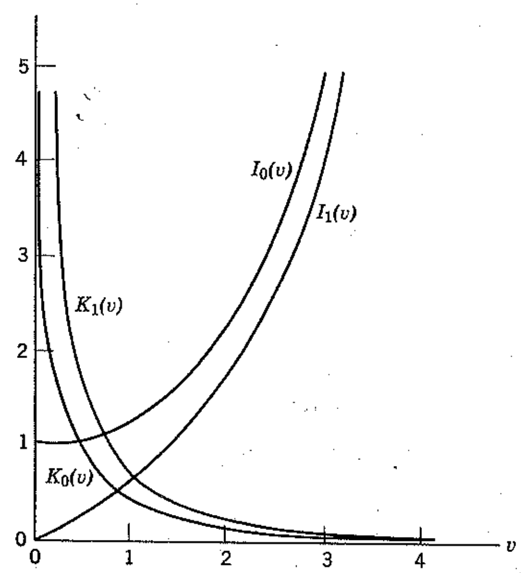


Fig. 3.26c Modified Bessel functions.

3.27 Bessel Function Formulas

Asymptotic Forms.

$$J_\nu(v) \underset{v\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi v}} \cos\left(v - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) \tag{1}$$

$$N_\nu(v) \underset{v\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi v}} \sin\left(v - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) \tag{2}$$

$$H_\nu^{(1)}(v) \underset{v\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi v}}\, e^{j[v-(\pi/4)-(\nu\pi/2)]} \tag{3}$$

$$H_\nu^{(2)}(v) \underset{v\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi v}}\, e^{-j[v-(\pi/4)-(\nu\pi/2)]} \tag{4}$$

$$j^{-\nu}J_\nu(jv) = I_\nu(v) \underset{v\to\infty}{\longrightarrow} \sqrt{\frac{1}{2\pi v}}\, e^{v} \tag{5}$$

$$j^{\nu+1}H_\nu^{(1)}(jv) = \frac{2}{\pi} K_\nu(v) \underset{v\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi v}}\, e^{-v} \tag{6}$$

Derivatives. The following formulas which may be found by differentiating the appropriate series, term by term, are valid for any of the

functions $J_\nu(v)$, $N_\nu(v)$, $H_\nu^{(1)}(v)$, $H_\nu^{(2)}(v)$. Let $R_\nu(v)$ denote any one of these, and R_ν' denote $(d/dv)[R_\nu(v)]$.

$\frac{d}{dv}$

$$R_0'(v) = -R_1(v) \tag{7}$$

$$R_1'(v) = R_0(v) - \frac{1}{v} R_1(v) \tag{8}$$

$$vR_\nu'(v) = \nu R_\nu(v) - vR_{\nu+1}(v) \tag{9}$$

$$vR_\nu'(v) = -\nu R_\nu(v) + vR_{\nu-1}(v) \tag{10}$$

$$\frac{d}{dv}[v^{-\nu}R_\nu(v)] = -v^{-\nu}R_{\nu+1}(v) \tag{11}$$

$$\frac{d}{dv}[v^{\nu}R_\nu(v)] = v^{\nu}R_{\nu-1}(v) \tag{12}$$

Note that

$\frac{d}{dv} R_0 =$ [illegible]

$$R_\nu'(Tr) = \frac{d}{d(Tr)}[R_\nu(Tr)] = \frac{1}{T}\frac{d}{dr}[R_\nu(Tr)]. \tag{13}$$

For the I and K functions different forms for the foregoing derivatives must be used. They may be obtained from these formulas by substituting Eqs. 3.26(16) and 3.26(17) in the preceding expressions. Some of these are

$$\begin{aligned} vI_\nu'(v) &= \nu I_\nu(v) + vI_{\nu+1}(v) \\ vI_\nu'(v) &= -\nu I_\nu(v) + vI_{\nu-1}(v) \end{aligned} \tag{14}$$

$$\begin{aligned} vK_\nu'(v) &= \nu K_\nu(v) - vK_{\nu+1}(v) \\ vK_\nu'(v) &= -\nu K_\nu(v) - vK_{\nu-1}(v) \end{aligned} \tag{15}$$

Recurrence Formulas. By recurrence formulas, it is possible to obtain the value for Bessel functions of any order, when the values of functions for any two other orders, differing from the first by integers, are known. For example, subtract (10) from (9). The result may be written

$$\frac{2\nu}{v} R_\nu(v) = R_{\nu+1}(v) + R_{\nu-1}(v). \tag{16}$$

As before, R_ν may denote J_ν, N_ν, $H_\nu^{(1)}$, $H_\nu^{(2)}$, but not I_ν or K_ν. For these, the recurrence formulas are

$$\frac{2\nu}{v} I_\nu(v) = I_{\nu-1}(v) - I_{\nu+1}(v) \tag{17}$$

$$\frac{2\nu}{v} K_\nu(v) = K_{\nu+1}(v) - K_{\nu-1}(v). \tag{18}$$

Integrals. Integrals that will be useful in solving later problems are given below. R_ν denotes J_ν, N_ν, $H_\nu^{(1)}$, or $H_\nu^{(2)}$:

$$\int v^{-\nu} R_{\nu+1}(v)\,dv = -v^{-\nu} R_\nu(v) \tag{19}$$

$$\int v^{\nu} R_{\nu-1}(v)\,dv = v^{\nu} R_\nu(v) \tag{20}$$

$$\int vR_\nu(\alpha v)R_\nu(\beta v)\,dv$$

$$= \frac{v}{\alpha^2-\beta^2}[\beta R_\nu(\alpha v)R_{\nu-1}(\beta v) - \alpha R_{\nu-1}(\alpha v)R_\nu(\beta v)], \qquad \alpha \neq \beta \tag{21}$$

$$\int vR_\nu^{\,2}(\alpha v)\,dv = \frac{v^2}{2}[R_\nu^{\,2}(\alpha v) - R_{\nu-1}(\alpha v)R_{\nu+1}(\alpha v)]$$

$$= \frac{v^2}{2}\left[R_\nu'^2(\alpha v) + \left(1 - \frac{\nu^2}{\alpha^2 v^2}\right)R_\nu^{\,2}(\alpha v)\right]. \tag{22}$$

3.28 Expansion of a Function as a Series of Bessel Functions

In Chapter 1 a study was made of the familiar method of Fourier series by which a function may be expressed over a given region as a series of sines or cosines. It is possible to evaluate the coefficients in such a case because of the orthogonality property of sinusoids, expressed in Art. 1.10. A study of the integrals, Eqs. 3.27(21) and 3.27(22), shows that there are similar orthogonality expressions for Bessel functions. For example, these integrals may be written for zero-order Bessel functions, and, if α and β are taken as p_m/a and p_q/a, where p_m and p_q are the mth and qth roots of $J_0(v) = 0$, that is, $J_0(p_m) = 0$ and $J_0(p_q) = 0$, $p_m \neq p_q$, then Eq. 3.27(21) gives

$$\int_0^a rJ_0\left(\frac{p_m r}{a}\right)J_0\left(\frac{p_q r}{a}\right)dr = 0. \tag{1}$$

So, if a function $f(r)$ may be expressed as an infinite sum of zero-order Bessel functions,

$$f(r) = b_1 J_0\left(p_1\frac{r}{a}\right) + b_2 J_0\left(p_2\frac{r}{a}\right) + b_3 J_0\left(p_3\frac{r}{a}\right) + \cdots$$

or

$$f(r) = \sum_{m=1}^{\infty} b_m J_0\left(\frac{p_m r}{a}\right). \tag{2}$$

The coefficients b_m may be evaluated in a manner similar to that used for Fourier coefficients by multiplying each term of (2) by $rJ_0(p_m r/a)$ and

integrating from 0 to a. Then by (1) all terms on the right disappear e the mth term:

$$\int_0^a rf(r)J_0\left(\frac{p_m r}{a}\right)dr = \int_0^a b_m r\left[J_0\left(\frac{p_m r}{a}\right)\right]^2 dr.$$

From Eq. 3.27(22),

$$\int_0^a rJ_0^{\,2}\left(\frac{p_m r}{a}\right)dr = \frac{a^2}{2}J_1^{\,2}(p_m).$$

So

$$\int_0^a rf(r)J_0\left(\frac{p_m r}{a}\right)dr = \frac{b_m a^2}{2}J_1^{\,2}(p_m)$$

or

$$b_m = \frac{2}{a^2 J_1^{\,2}(p_m)}\int_0^a rf(r)J_0\left(\frac{p_m r}{a}\right)dr.$$

Thus a formula for the coefficients of the series (2) is derive mathematical study of the subject would be concerned with showin the series thus formally derived actually does converge to the d function over the range of interest. Such a discussion is outside the of this text, but the results of such studies show that completenes convergence requirements are met, so that such a series may be us represent any piecewise continuous function over the range $0 < r <$

Problems

3.28a Write a function $f(r)$ in terms of nth order Bessel functions ov range 0 to a and determine the coefficients.

3.28b Determine coefficients for a function $f(r)$ expressed over the rang a as a series of zero-order Bessel functions as follows:

$$f(r) = \sum_{m=1}^{\infty} c_m J_0\left(\frac{p_m' r}{a}\right)$$

where p_m' denotes the mth root of $J_0'(v) = 0$ [i.e., $J_1(v) = 0$].

3.29 Fields Described by Cylindrical Harmonics

We will consider here the two basic types of boundary value prob which exist in axially symmetric cylindrical systems. These can be u stood by reference to Fig. 3.29a. In one type both Φ_1 and Φ_2, the poter on the ends, are zero and a non-zero potential Φ_3 is applied to the c drical surface. In the second type $\Phi_3 = 0$ and either (or both) Φ_1 o are non-zero. The gaps between ends and side are considered negli