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Problem Set 4

3.2
The potential everywhere satisfies the equation

$$\nabla^2 \Phi = 0$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)\Phi + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right)\Phi = 0$$

We can write

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right)\Phi = -l(l+1)\Phi$$
$$\Phi = \sum_{l=0}^{\infty} A_l(r) P_l(\cos\theta)$$

Then substituting back into the Laplace equation, obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)A_l(r) + \frac{-l(l+1)}{r^2}A_l(r) = 0$$

This has the solutions

$$A_l(r) = B_l r^k$$

Using this solution, obtain the indicial equation

$$k(k-1) = -l(l+1) \Rightarrow k = l, -(l+1)$$

So, obtain the general solution everywhere

$$\Phi = \sum_{l=0}^{\infty}\left(B_l r^l + A_l r^{-(l+1)}\right)P_l(\cos\theta)$$

a.)
Demand that the potential be finite at the origin and at infinity. Thus, we have the potential divided into two regions, inside the sphere and outside the sphere:

$$\Phi_{in} = \sum_{l=0}^{\infty} B_l r^l P_l(\cos\theta) \qquad \Phi_{out} = \sum_{l=0}^{\infty} A_l r^{-(l+1)} P_l(\cos\theta)$$

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3.2
a.) cont'd

Across the surface of the sphere, we have the conditions

$$\left.\frac{\partial \Phi_{in}}{\partial r}\right|_{r=R} - \left.\frac{\partial \Phi_{out}}{\partial r}\right|_{r=R} = \frac{\sigma}{\varepsilon_0}$$
$$\Phi_{in}\big|_R = \Phi_{out}\big|_R \Rightarrow A_l = B_l R^{2l+1}$$

So we can rewrite the potential outside the sphere as

$$\Phi_{out} = \sum_{l=0}^{\infty} B_l \frac{R^{2l+1}}{r^{l+1}} P_l(\cos\theta)$$

The charge density on the surface of the sphere is

$$\sigma = \begin{cases} 0 & 0 \le \theta < \alpha \\ \dfrac{Q}{4\pi R^2} & \alpha \le \theta \le \pi \end{cases} = \sum_{l=0}^{\infty} D_l P_l(\cos\theta)$$

From this, we find

$$\begin{aligned} D_l &= \frac{2l+1}{2}\int_{-1}^{1} \sigma P_l(\cos\theta)\, d(\cos\theta) \\ &= \frac{2l+1}{2}\int_{-1}^{\cos\alpha} \frac{Q}{4\pi R^2} P_l(\cos\theta)\, d(\cos\theta) \\ &= \frac{Q}{8\pi R^2}\int_{-1}^{\cos\alpha} \left[P_{l+1}{}'(\cos\theta) - P_{l-1}{}'(\cos\theta)\right] d(\cos\theta) \\ &= \frac{Q}{8\pi R^2}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] \end{aligned}$$

Then, substituting, obtain

$$\left.\frac{\partial \Phi_{in}}{\partial r}\right|_R = \sum_{l=0}^{\infty} l B_l R^{l-1} P_l(\cos\theta)$$
$$\left.\frac{\partial \Phi_{out}}{\partial r}\right|_R = \sum_{l=0}^{\infty} -(l+1) B_l \frac{R^{2l+1}}{R^{l+2}} P_l(\cos\theta) = -\sum_{l=0}^{\infty} (l+1) B_l R^{l-1} P_l(\cos\theta)$$

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3.2 cont'd
a.) cont'd

Substitution then yields

$$(2l+1)B_l R^{l-1} = \frac{Q}{8\pi R^2 \varepsilon_0}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]$$

$$B_l = \frac{Q}{8\pi(2l+1)R^{l+1}\varepsilon_0}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]$$

Finally obtain the potentials

$$\begin{aligned}\Phi_{in} &= \sum_{l=0}^{\infty} \frac{Q}{8\pi(2l+1)R^{l+1}\varepsilon_0}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] r^l P_l(\cos\theta) \\ &= \frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{r^l}{R^{l+1}}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]P_l(\cos\theta)\end{aligned}$$

$$\begin{aligned}\Phi_{out} &= \sum_{l=0}^{\infty} \frac{Q}{8\pi(2l+1)R^{l+1}\varepsilon_0}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]\frac{R^{2l+1}}{r^{l+1}} P_l(\cos\theta) \\ &= \frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{R^l}{r^{l+1}}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]P_l(\cos\theta)\end{aligned}$$

b.)

Find the electric field inside the sphere at the origin:

$$\vec{E}_{in}\Big|_{r=0} = -\vec{\nabla}\Phi_{in}\Big|_{r=0} = -\frac{\partial\Phi_{in}}{\partial r}\bigg|_{r=0}\hat{r} - \left(\frac{1}{r}\frac{\partial\Phi_{in}}{\partial\theta}\right)\bigg|_{r=0}\hat{\theta}$$

$$\begin{aligned}\frac{\partial\Phi_{in}}{\partial r}\bigg|_{r=0} &= \frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\, l\,\frac{r^{l-1}}{R^{l+1}}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]P_l(\cos\theta)\bigg|_{r=0} \\ &= \frac{Q}{24\pi\varepsilon_0 R^2}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right]P_1(\cos\theta) \\ &= \frac{Q}{16\pi\varepsilon_0 R^2}\left[\cos^2\alpha - 1\right]\cos\theta = -\frac{Q}{16\pi\varepsilon_0 R^2}(\sin^2\alpha)\cos\theta\end{aligned}$$

$$\frac{1}{\sin\theta}\frac{\partial\Phi_{out}}{\partial\theta}\bigg|_{r=0} = -\frac{\partial\Phi_{out}}{\partial(\cos\theta)}\bigg|_{r=0} = -\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{r^l}{R^{l+1}}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]P_l'(\cos\theta)\bigg|_{r=0} = 0$$

Problem Set 4

3.2 cont'd
b.) cont'd

So then find the electric field at the origin to be

$$\vec{E}_{in}\Big|_{r=0} = \frac{Q}{16\pi\varepsilon_0 R^2}\left(\sin^2\alpha\right)\cos\theta\hat{r}$$

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3.5

To solve this problem, we use equations 3.38, 3.62, and 3.70 in Jackson

$$\Phi(\vec{x}) = \frac{a\left(a^2 - r^2\right)}{4\pi} \int \frac{V\left(\theta',\phi'\right)}{\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} d\Omega'$$

$$\frac{1}{|\vec{x} - a\hat{r}|} = \sum_{l=0}^{\infty} \frac{r^l}{a^{l+1}} P_l\left(\cos\gamma\right) = \frac{1}{\left(r^2 + a^2 - 2ar\cos\gamma\right)^{1/2}}$$

$$\sum_{l=0}^{\infty} \frac{r^l}{a^{l+1}} P_l'\left(\cos\gamma\right) = \frac{ar}{\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} = \sum_{l=1}^{\infty} \frac{r^l}{a^{l+1}} P_l'\left(\cos\gamma\right)$$

Substituting, we then obtain

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{a\left(a^2 - r^2\right)}{4\pi} \int \sum_{l=1}^{\infty} \frac{r^l}{a^{l+1}} P_l'\left(\cos\gamma\right) V\left(\theta',\phi'\right) d\Omega' \\
&= \frac{1}{4\pi} \sum_{l=1}^{\infty} \int \left[\left(\frac{r}{a}\right)^{l-1} - \left(\frac{r}{a}\right)^{l+1} \right] P_l'\left(\cos\theta\right) V\left(\theta',\phi'\right) d\Omega' \\
&= \frac{1}{4\pi} \int \left[\sum_{l=0}^{\infty} \left(\frac{r}{a}\right)^{l} P_{l+1}'\left(\cos\theta\right) - \sum_{l=2}^{\infty} \left(\frac{r}{a}\right)^{l} P_{l-1}'\left(\cos\theta\right) \right] V\left(\theta',\phi'\right) d\Omega' \\
&= \frac{1}{4\pi} \int \left[1 + \sum_{l=1}^{\infty} \left(\frac{r}{a}\right)^{l} \left\{ P_{l+1}'\left(\cos\theta\right) - P_{l-1}'\left(\cos\theta\right) \right\} \right] V\left(\theta',\phi'\right) d\Omega' \\
&= \frac{1}{4\pi} \int \left\{ 1 + \sum_{l=1}^{\infty} \left(\frac{r}{a}\right)^{l} (2l+1) P_l\left(\cos\theta\right) \right\} V\left(\theta',\phi'\right) d\Omega' \\
&= \frac{1}{4\pi} \int \sum_{l=0}^{\infty} \left(\frac{r}{a}\right)^{l} (2l+1) P_l\left(\cos\theta\right) V\left(\theta',\phi'\right) d\Omega'
\end{aligned}$$

Using

$$P_l\left(\cos\theta\right) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*\left(\theta',\phi'\right) Y_{lm}\left(\theta,\phi\right)$$

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3.5 cont'd
we finally obtain

$$\Phi(\vec{x}) = \int \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{r}{a}\right)^{l} Y_{lm}^{*}(\theta', \phi') Y_{lm}(\theta, \phi) V(\theta', \phi') d\Omega'$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{r}{a}\right)^{l} Y_{lm}(\theta, \phi) \int Y_{lm}^{*}(\theta', \phi') V(\theta', \phi') d\Omega'$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{r}{a}\right)^{l} Y_{lm}(\theta, \phi) A_{lm}$$

where

$$A_{lm} = \int Y_{lm}^{*}(\theta', \phi') V(\theta', \phi') d\Omega'$$

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3.16
b.)

The Green's function

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$$

satisfies the equation

$$\nabla^2 G = -4\pi\delta(\vec{r}-\vec{r}')$$
$$\nabla^2 G = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')$$

We have that the delta functions in ρ and ϕ can be written as

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$
$$\frac{1}{\rho}\delta(\rho-\rho') = \int_0^\infty kJ_m(k\rho)J_m(k\rho')\,dk$$

So, we can write the Green's function as

$$G(\vec{r},\vec{r}') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty ke^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')g_{km}(z,z')\,dk$$

Substituting back into the equation for the Green's function, obtain the equation for g:

$$\left(\frac{\partial^2}{\partial z^2}-k^2\right)g_{km}(z,z') = -4\pi\delta(z-z')$$

From this, we can write the solution

$$g_{km}(z,z') = A_{km}e^{-k(z_>-z_<)}$$

Integrating the equation for g across the boundary z=z', obtain

$$A_{km}(-k) - A_{km}k = -4\pi \Rightarrow A_{km} = \frac{2\pi}{k}$$

Substituting back into the Green's function, we finally obtain

$$\frac{1}{|\vec{r}-\vec{r}'|} = G(\vec{r},\vec{r}') = \sum_{m=-\infty}^{\infty}\int_0^\infty e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')e^{-k(z_>-z_<)}\,dk$$

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3.16 cont'd
d.)

We have the relation

$$e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m\left(k\rho\right)$$

Using orthogonality, we obtain

$$\int_0^{2\pi} e^{ik\rho\cos\phi} e^{-im'\phi} d\phi = 2\pi i^{m'} J_{m'}\left(k\rho\right)$$

$$J_{m'}\left(k\rho\right) = \frac{1}{2\pi i^{m'}} \int_0^{2\pi} e^{ik\rho\cos\phi} e^{-im'\phi} d\phi$$

Then, we can make the substitution

$$x = k\rho$$

$$J_{m'}(x) = \frac{1}{2\pi i^{m'}} \int_0^{2\pi} e^{ix\cos\phi} e^{-im'\phi} d\phi$$

3.17 $\nabla^2 G = \delta(\underline{r}-\underline{r}') = \frac{\delta(r-r')}{r}\delta(\phi-\phi')\delta(z-z')$

In cylindrical coordinates (r,φ,z)

with $G=0$ at $z=0$ and $z=L$ Try

$$G = \sum_{nm'} G_{nm'}(r)\, e^{im'\phi}\sin(k_n z)$$

where $k_n = n\pi/L$

Insert above, multiply by $e^{-im\phi}\sin(k_n z)\,dz\,d\phi$ and integrate

$$\frac{L}{2}\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{m^2}{r^2} - k_n^2\right]G_{nm} = \frac{\delta(r-r')}{r}\, e^{-im\phi'}\sin(k_n z')$$

LET $\quad G_{nm} = e^{-im\phi'}\sin(k_n z')\,\tilde{G}_{nm}$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{m^2}{r^2} - k_n^2\right]\tilde{G}_{nm}(r) = \frac{2\,\delta(r-r')}{rL}$$

FOR $r>r'$ $\quad \tilde{G}_{nm}(r) = A\,K_m(k_n r)$

$r<r'$ $\quad \tilde{G}_{nm}(r) = B\,I_m(k_n r)$

continuity of $\tilde{G}_{nm}$ at $r' \Rightarrow \quad A\,K_m(k_n r') = B\,I_m(k_n r')$

$$\int_{r'-\epsilon}^{r'+\epsilon} dr\, r\,(\text{ABOVE}) \Rightarrow \quad r\frac{d}{dr}\tilde{G}_{nm}\Big|_{r'-\epsilon}^{r'+\epsilon} = 2/L$$

$$k_n r'\left[A K_m'(k_n r') - B I_m'(k_n r')\right] = 2/L$$

eliminate B

$$A\,(k_n r')\left[K_m'(k_n r') - \frac{K_m I_m'}{I_m}\right] = 2/L$$

$$A = \frac{2}{zL} I_m(z) \Big/ \left[I_m K_m' - I_m' K_m\right] \qquad z = k_n r'$$

Wronskian Relation for modified Bessel Functions

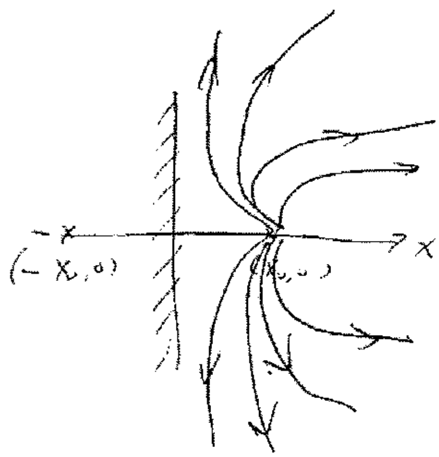
$$I_m' K_m - I_m K_m' = 1/Z$$

$$A = -\frac{2}{L} I_m(k_n r') \qquad B = -\frac{2}{L} K_m(k_n r')$$

THUS $$G_{nm} = -\frac{4}{L} \sum_{nm} I_m(k_n r_<) K_m(k_n r_>) \sin k_n z \sin k_n z' \, e^{im(\phi-\phi')}$$

mystery minus

3 A.



Without boundary (infinite sheet) a current flows into the sheet would spread out radially symmetrically

$$\therefore \vec{J} = \frac{I}{2\pi\rho}\hat{\rho}$$

with the semi-infinite sheet, or a boundary at $x=0$, we have the b.c. $\hat{n}\cdot\vec{J} = 0$ at $x=0$

but $\vec{J} = \sigma\vec{E} = -\sigma\nabla\phi$

$$\therefore \hat{n}\cdot\vec{J} = 0 \Rightarrow \left.\frac{\partial\phi}{\partial x}\right|_{x=0} = 0$$

We can put an image current I at $\left(\begin{matrix} x=-X_0 \\ y=0 \end{matrix}\right)$

$$\therefore \vec{E} = -\nabla\phi$$

$$x>0, \quad \phi = -\frac{I}{\sigma 2\pi}\left[\ln\sqrt{(x+x_0)^2+y^2} + \ln\sqrt{(x-x_0)^2+y^2}\right]$$

$$x>0 \quad \vec{E} = \frac{I}{\sigma 2\pi}\left[\left(\frac{x+x_0}{(x+x_0)^2+y^2} + \frac{x-x_0}{(x-x_0)^2+y^2}\right)\hat{i} + \left(\frac{y}{(x+x_0)^2+y^2} + \frac{y}{(x-x_0)^2+y^2}\right)\hat{j}\right]$$

$$\vec{J} = \sigma\vec{E}$$