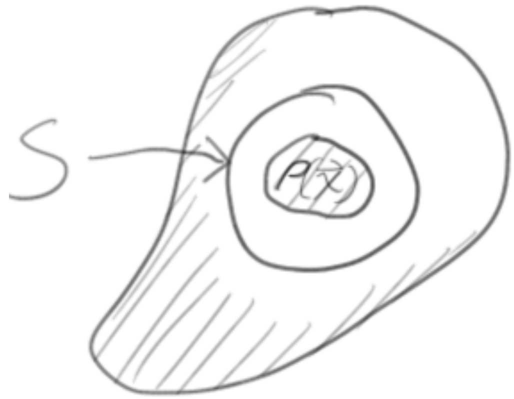
ENEE680 Electromagnetic Theory I

Hmwk Set 1

1.1
a.)

Suppose we have a conductor and place some charge density within it.

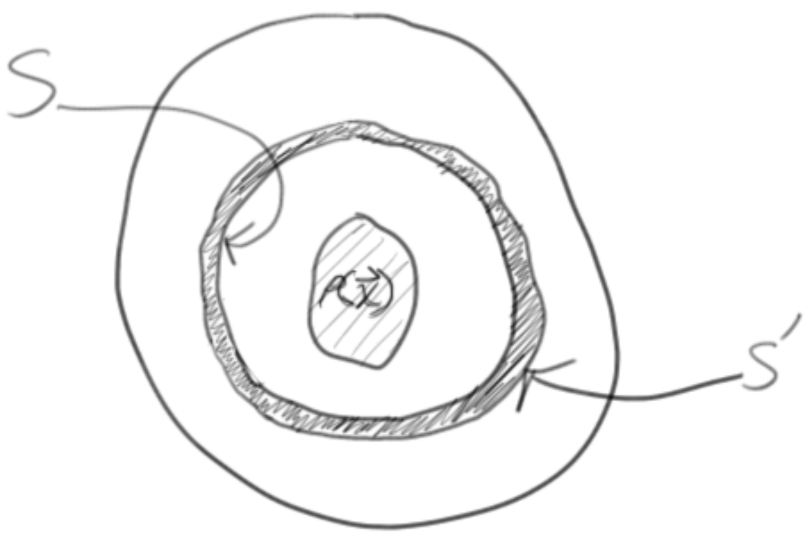
$$Q = \int_V d^3\vec{x}'\rho(\vec{x}')$$



For a surface S inside the conductor enclose the charge density

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\int_V d^3\vec{x}'\rho(\vec{x}')$$

This will cause an electric field that will pull free charges in the conductor to move to cancel the total charge Q from the charge density. Once the total charge is cancelled, there is no more field through S. However, because the conductor is neutral, pulling a total charge -Q to cancel the charge density will cause a charge Q to develop in the layer immediately surrounding S. Call this layer S', and let it be infinitesimally above the surface S.



ENEE680 Electromagnetic Theory I

Hmwk Set 1

1.1 cont'd
a.) cont'd

At S we have

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0} - \frac{Q}{\varepsilon_0} = 0$$

while at S'

$$\oint_{S'} \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0}$$

This process will continue until we reach the surface of the conductor, at which point there is no more charge available to move. Thus the total charge Q ends up distributed on the surface of the conductor.

We also notice that as a result of the process

$$\oint_S \vec{E} \cdot d\vec{a} = 0$$

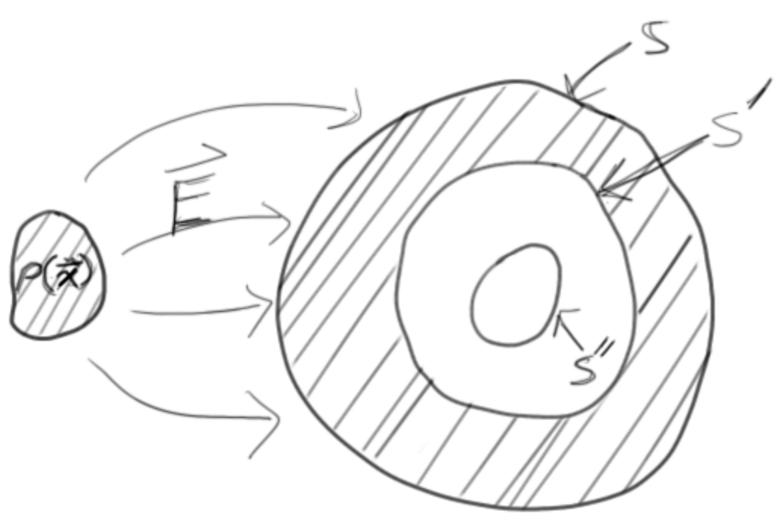
for any surface S inside the conductor. Thus,

$$\vec{E} = 0$$

everywhere inside the conductor.

b.)

Suppose we have a hollow conductor and there is a charge distribution situated outside the conductor:



Hmwk Set 1

1.1 cont'd
b.) cont'd

From part a, we know that the charges in the conductor will move until $\vec{E} = 0$ every-where in the volume of the conductor (the region between S and S'). For any arbitrary closed surface S" in the hollow

$$\oint_{S''} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} \int_{V''} d^3\vec{x}' \rho(\vec{x}') = 0$$

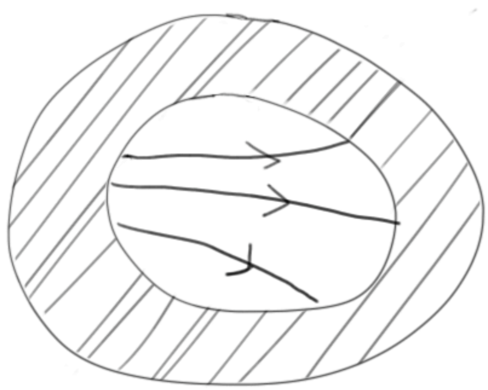
because everywhere in the hollow

$$\rho(\vec{x}') = 0$$

Thus,

$$\oint_{S''} \vec{E} \cdot d\vec{a} = 0 = \int_{V''} \vec{\nabla} \cdot \vec{E} \; d^3\vec{x}'$$

meaning that no field lines can start or terminate within the volume of the hollow. So, field lines must pass all the way through in continuous paths.



Because we have an electrostatic situation

$$\oint_{C} \vec{E} \cdot d\vec{l} = 0$$

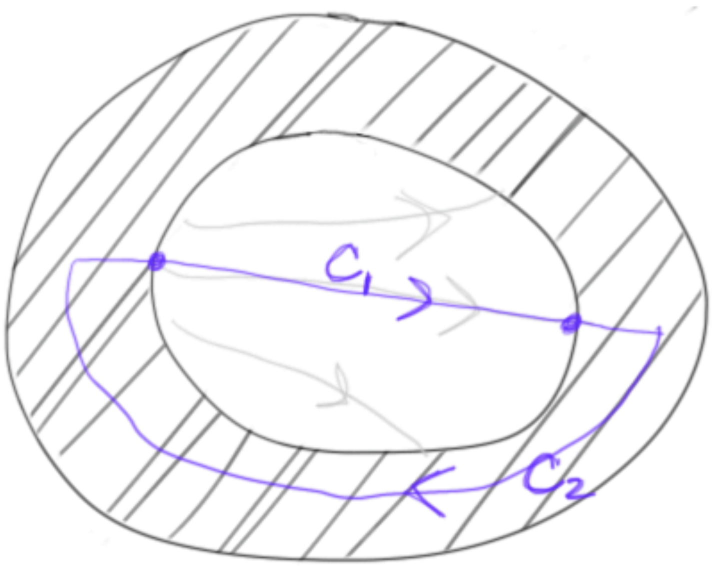
for any arbitrary closed path C.

ENEE680 Electromagnetic Theory I

Hmwk Set 1

1.1 cont'd
b.) cont'd

Suppose choose C such part of it runs through the conductor and the other part runs through the hollow, parallel to any arbitrary field line:



$$\oint_C \vec{E}\cdot d\vec{l} = 0 = \int_{C_2} \vec{E}\cdot d\vec{l} + \int_{C_1} \vec{E}\cdot d\vec{l}$$

Because $\vec{E} = 0$ everywhere inside the conductor

$$\int_{C_2} \vec{E}\cdot d\vec{l} = 0$$

Thus, for any continuous contour through the hollow

$$\int_{C_1} \vec{E}\cdot d\vec{l} = E_0 L = 0 \Rightarrow E_0 = 0$$

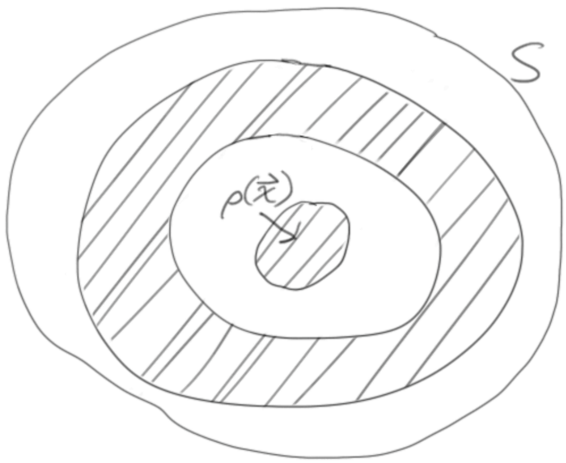
since L≠0. Therefore, the electric field is zero everywhere inside the hollow.

ENEE680 Electromagnetic Theory I

Hmwk Set 1

1.1 cont'd
b.) cont'd

If a charge density is place inside the hollow



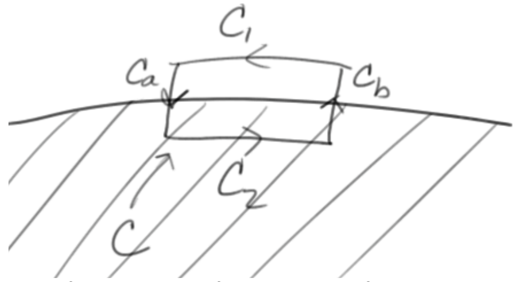
then by Gauss's Law, for any surface outside the conductor

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\int_V d^3\vec{x}'\rho(\vec{x}') \neq 0$$

So there must be an electric field outside the conductor.

c.)

At the surface of a conductor, we can place an arbitrarily small loop parallel to the surface



$$\oint_C \vec{E}\cdot d\vec{l} = 0 = \int_{C_2}\vec{E}\cdot d\vec{l} + \int_{C_1}\vec{E}\cdot d\vec{l} + \int_{C_a}\vec{E}\cdot d\vec{l} + \int_{C_b}\vec{E}\cdot d\vec{l}$$

We shrink the curves C_a and C_b such they are arbitrarily small compared to C_1 and C_2. Then

$$\oint_C \vec{E}\cdot d\vec{l} = 0 = \int_{C_2}\vec{E}_{in}\cdot\hat{t}dl + \int_{C_1}\vec{E}_{out}\cdot\hat{t}dl$$

ENEE680 Electromagnetic Theory I

Hmwk Set 1

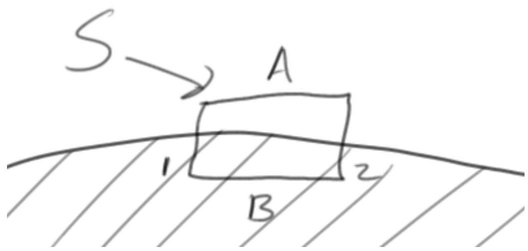
1.1 cont'd
c.) cont'd

However, inside the conductor $\vec{E}_{in} = 0$. Thus,

$$\int_{C_1} \vec{E}_{out} \cdot \hat{t} dl = 0$$

So there is no component of the electric field tangential to the conductor's outer surface. This leaves only the component normal to the outer surface

Now if we choose a closed surface S intersecting the conductor's surface.



Further, we choose the surface small enough so that the surface charge density σ is arbitrarily close to being uniform.

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} \int_V d^3\vec{x}' \sigma(\vec{x}')$$

$$\oint_S \vec{E} \cdot d\vec{a} = \int_A \vec{E} \cdot d\vec{a} + \int_B \vec{E} \cdot d\vec{a} + \int_1 \vec{E} \cdot d\vec{a} + \int_2 \vec{E} \cdot d\vec{a}$$

Since the electric field is zero everywhere inside the conductor

$$\int_B \vec{E} \cdot d\vec{a} = 0$$

Thus, we have

$$\oint_S \vec{E} \cdot d\vec{a} = \int_A \vec{E} \cdot d\vec{a} + \int_1 \vec{E} \cdot d\vec{a} + \int_2 \vec{E} \cdot d\vec{a}$$

We can shift and deform the closed surface such side A is arbitrarily close and parallel to the conductor's surface, submerging sides 1 and 2 in the conductor. So,

$$\int_1 \vec{E} \cdot d\vec{a} = 0 = \int_2 \vec{E} \cdot d\vec{a}$$

$$\oint_S \vec{E} \cdot d\vec{a} = \int_A \vec{E} \cdot d\vec{a} = E_n A = \frac{1}{\varepsilon_0} \int_A dA\ \sigma = \frac{\sigma}{\varepsilon_0} A \Rightarrow E_n = \frac{\sigma}{\varepsilon_0}$$

ENEE680 Electromagnetic Theory I

Hmwk Set 1

1.5

Given the potential

$$\Phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)$$

From the Poisson equation

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}$$

we find

$$\rho = -\varepsilon_0\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right)\Phi(\vec{r}) = -\frac{\alpha^3 e^{-\alpha r}}{8\pi}q$$

However, as $r \to 0$, $\Phi(\vec{r}) \to \infty$. This means there is a point charge at r=0. If we choose an arbitrarily small sphere of radius r around r=0, the potential over that surface can be written

$$\Phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(1+\frac{\alpha r}{2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}+\frac{\alpha}{2}\right)$$

Then we can use Gauss's Law over the surface to find the enclosing charge density

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\int_V d^3\vec{r}\ \rho_0(\vec{r}) \qquad \vec{E} = -\vec{\nabla}\Phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$\frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V d^3\vec{r}\ \rho_0(\vec{r}) \Rightarrow \rho_0(\vec{r}) = q\delta(\vec{r})$$

Thus, we find the total charge density as

$$\rho(\vec{r}) = q\delta(\vec{r}) - \frac{\alpha^3 e^{-\alpha r}}{8\pi}q$$

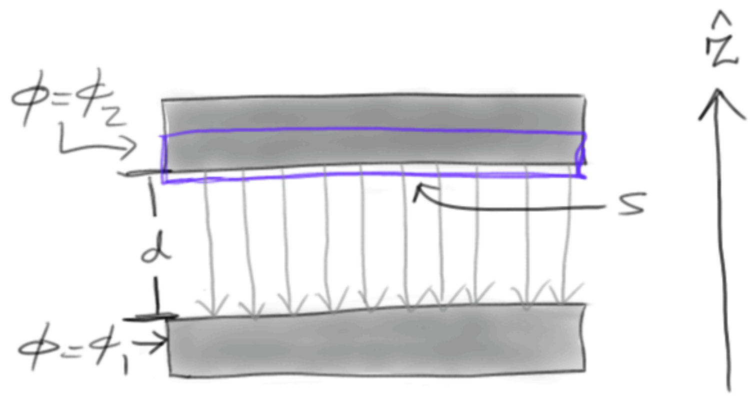
ENEE680 Electromagnetic Theory I

Hmwk Set 1

1.6
Capacitance is defined by

$$C = \frac{Q}{\Delta\phi}$$

a.)



The electric field between the plates is

$$\vec{E} = -\frac{Q}{\varepsilon_0 A}\hat{z}$$

The potential difference is found on a straight path in the z-direction as

$$\frac{Q}{C} = \Delta\phi = \int_0^d \vec{E}\cdot d\vec{l} = \frac{Qd}{\varepsilon_0 A}$$

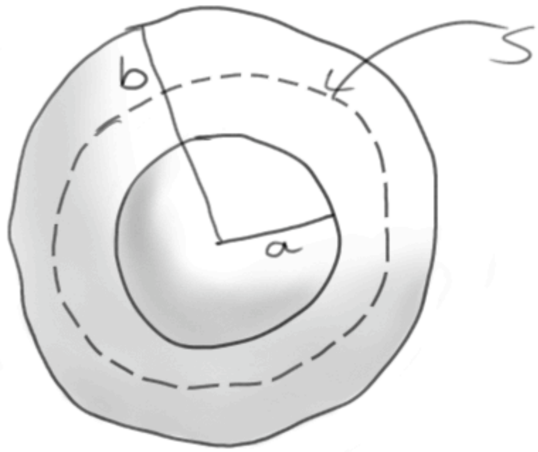
Thus, we find the capacitance

$$C = \frac{\varepsilon_0 A}{d}$$

ENEE680 Electromagnetic Theory I

Hmwk Set 1

1.6 cont'd
b.)



Between the spheres, create the spherical surface S with radius r, b < r < a. The electric field at the surface is

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

The potential difference along a path from the surface of the inner sphere to the interior surface of the outer sphere is given by

$$\Delta\phi = \int_{r=a}^{r=b} \vec{E}\cdot d\vec{l} = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{(b-a)}{ab}$$

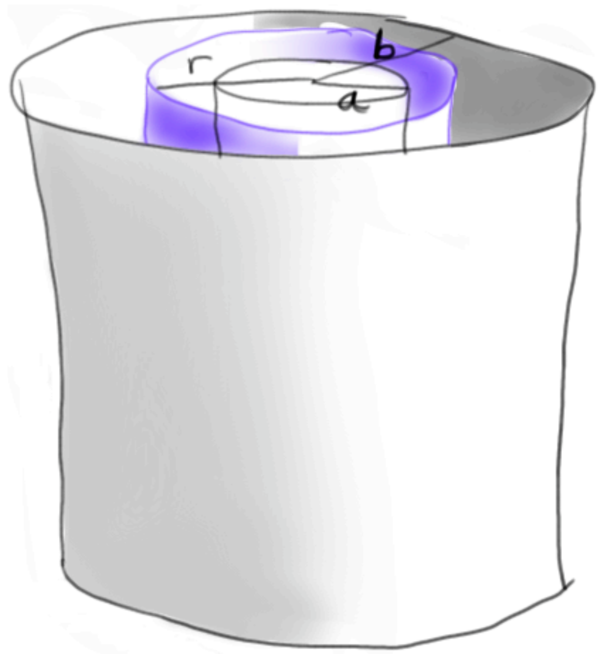
Then, by substitution of the definition of capacitance, we find

$$\frac{Q}{C} = \frac{Q}{4\pi\varepsilon_0}\frac{(b-a)}{ab} \Rightarrow C = 4\pi\varepsilon_0\frac{ab}{(b-a)}$$

ENEE680 Electromagnetic Theory I

Hmwk Set 1

1.6 cont'd
c.)



The electric field between the plates is given by

$$\vec{E} = \frac{Q}{2\pi\varepsilon_0 L r}\hat{r}$$

Again find the potential difference along a straight path from the inner surface to the outer surface

$$\Delta\phi = \int_{r=a}^{r=b} \vec{E}\cdot d\vec{l} = \frac{Q}{2\pi\varepsilon_0 L}\int_a^b \frac{1}{r}dr = \frac{Q}{2\pi\varepsilon_0 L}\left(\ln b - \ln a\right) = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$

So find the capacitance

$$\frac{Q}{C} = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right) \Rightarrow C = \frac{2\pi\varepsilon_0 L}{\ln\left(b/a\right)}$$

ENEE680 Electromagnetic Theory I

Hmwk Set 1

1.12

Apply Green's Theorem to two potentials Φ and Φ' :

$$\int d^3\vec{x}\left(\Phi'\nabla^2\Phi-\Phi\nabla^2\Phi'\right)=\oint_S\left(\Phi'\frac{\partial\Phi}{\partial n}-\Phi\frac{\partial\Phi'}{\partial n}\right)da$$

Substitute Poisson's equation and continuity across surface boundaries

$$\nabla^2\Phi=-\frac{\rho}{\varepsilon_0}\qquad\frac{\partial\Phi}{\partial n}=\frac{\sigma}{\varepsilon_0}\qquad\qquad\nabla^2\Phi'=-\frac{\rho'}{\varepsilon_0}\qquad\frac{\partial\Phi'}{\partial n}=\frac{\sigma'}{\varepsilon_0}$$

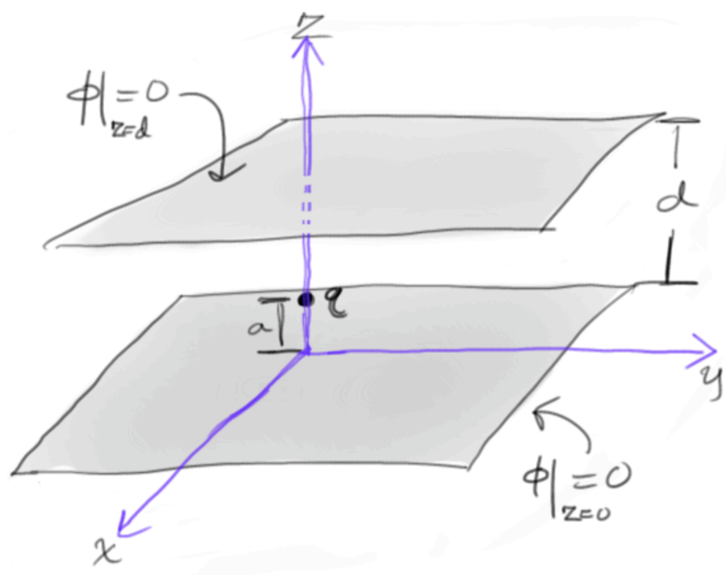
to find

$$\int d^3\vec{x}\left(-\Phi'\frac{\rho}{\varepsilon_0}+\Phi\frac{\rho'}{\varepsilon_0}\right)=\oint_S\left(\Phi'\frac{\sigma}{\varepsilon_0}-\Phi\frac{\sigma}{\varepsilon_0}\right)da$$

$$\int d^3\vec{x}\ \Phi\rho'+\oint_S\Phi\sigma' da=\int d^3\vec{x}\ \Phi'\rho+\oint_S\Phi'\sigma da$$

1.13

Consider a point charge q between the two grounded infinite conductive plates. A surface charge σ is induced on the plates as a result of q, and the potential between the plates is given by ϕ:



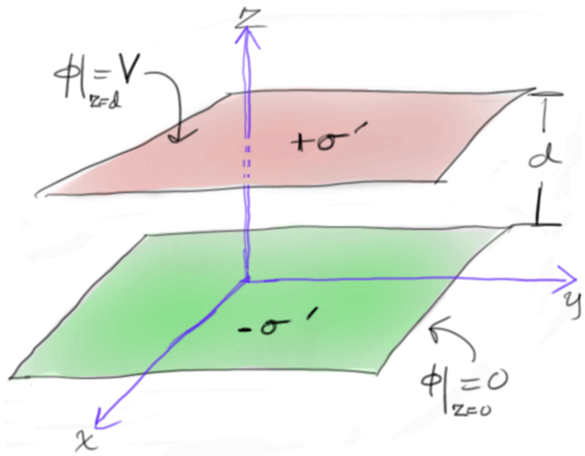
$$\rho(\vec{r})=q\delta(x)\delta(y)\delta(z-a)$$
$$\phi(z=0)=0\quad\phi(z=d)=0$$

ENEE680 Electromagnetic Theory I

Hmwk Set 1

1.13 cont'd

If we introduce a corresponding situation in which two infinite conducting plates have opposite surface charge densities σ' and -σ', giving rise to the potential ψ between the plates:



$$\rho'(\vec{r}) = 0 \quad \sigma' = \frac{\varepsilon_0 V}{d}$$
$$\psi'(z) = \frac{Vz}{d}$$

then we can use Green's Reciprocation Theorem as follows to find the total charge induced in the plate of the first situation:

$$\int d^3\vec{x}\ \rho'\phi + \oint_S da\ \sigma'\phi = \int d^3\vec{x}\ \rho\psi' + \oint_S da\ \sigma\psi'$$
$$0 = \int d^3\vec{x}\ \rho\psi' + \oint_S da\ \sigma\psi'$$
$$0 = q\frac{Va}{d} + V\oint_S da\ \sigma = q\frac{Va}{d} + VQ \Rightarrow Q = -q\frac{a}{d}$$

ENEE680 Electromagnetic Theory I

Hmwk Set 1

1A

Suppose apply the voltage V to plate at z=d and ground the plate at z=0. A charge density flows between the plates in the negative z-direction at nearly the speed of light.

$$\vec{J} = \rho c\hat{z} = -J\hat{z}$$

Using Poisson's equation

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_0} = \frac{J}{c\varepsilon_0}$$

The potential is symmetric in the x and y directions. So,

$$\partial_x \to 0 \qquad \partial_y \to 0$$

Thus, obtain

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial z^2} = \frac{J}{c\varepsilon_0} \Rightarrow \Phi = \frac{J}{2c\varepsilon_0}z^2$$

This leads to the relation of the applied potential and the current density

$$V = \frac{J}{2c\varepsilon_0}d^2 \Rightarrow J = \frac{2c\varepsilon_0 V}{d^2}$$

1B

i)

In 1-D, the equation for the Green's function is

$$\frac{d^2}{dx^2}G(x,x') = -\delta(x-x')$$

with boundary conditions

$$G(x,0) = 0 \quad G(x,b) = 0$$

Away from x=x', we have

$$\frac{d^2}{dx^2}G(x,x') = 0$$

which quickly leads to solution

$$G(x,x') = Ax_<(x_> - b)$$

ENEE680 Electromagnetic Theory I

Hmwk Set 1

1B cont'd

Match across x=x'

$$\int_{x-\varepsilon}^{x+\varepsilon} dx' \ \frac{d^2}{dx'^2} G(x,x') = -\int_{x-\varepsilon}^{x+\varepsilon} dx' \ \delta(x-x')$$

$$\left.\left.\frac{d}{dx'}G(x,x')\right|_{x>x'}\right|_{x'=x} - \left.\left.\frac{d}{dx'}G(x,x')\right|_{x>x'}\right|_{x'=x} = -1$$

$$Ax - A(x-b) = -1 \Rightarrow A = -\frac{1}{b}$$

Thus find

$$G(x,x') = -\frac{1}{b}x_<(x_> - b)$$

***ii*)**

Use Green's Theorem in 1-D

$$\int_0^b \left[\Phi\nabla'^2 G - \left(\nabla'^2\Phi\right)G\right]dx' = \left[\Phi\frac{\partial G}{\partial x'} - G\frac{\partial \phi}{\partial x'}\right]\Bigg|_0^b$$

Applying boundary conditions, we obtain

$$-\Phi(x) + \int_0^b \frac{\rho}{\varepsilon_0}G(x,x')dx' = \Phi\left.\frac{\partial G}{\partial x'}\right|_0^b = -\frac{1}{b}x\Phi(b) + \frac{(x-b)}{b}\Phi(0)$$

$$\Phi(x) = -\frac{1}{b\varepsilon_0}\int_0^b \rho(x')x_<(x_> - b)dx' + \frac{x}{b}\Phi(b) - \frac{(x-b)}{b}\Phi(0)$$

1.C) The potential surrounding a point charge, q, located at position $\mathbf{x}_0$ in a plasma is given by,

$$\Phi_0(\mathbf{x}) = \frac{q \exp(-|\mathbf{x}-\mathbf{x}_0| / \lambda_d)}{4\pi\varepsilon_0 |\mathbf{x}-\mathbf{x}_0|} ,$$

where λ_d is known as the Debye length. a) Find the charge density induced in the plasma. b) Show that it is proportional to the local potential.

1) $\nabla^2 \phi_0 = -(\rho_{free} + \rho_{ind})/\epsilon_0$

$\rho_{free} = q\,\delta(\underset{\sim}{r}) \qquad \underset{\sim}{r} = \underline{x} - \underline{x}_0$

For $r > 0$ $\qquad \rho_{ind}/\epsilon_0 = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \phi_0}{\partial r}$

$$\rho_{ind} = -\frac{q}{4\pi\lambda^2} \exp(-r/\lambda) = -\frac{1}{4\pi\lambda^2}\Phi_0$$

$$E_r = -\frac{\partial \Phi_0}{\partial r} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r^2} + \frac{1}{r\lambda}\right)\exp(-r/\lambda)$$