

VECTOR ANALYSIS

Dot-product: $\vec{A} \cdot \vec{B} \equiv AB \cos \mathbf{q}_{AB}$

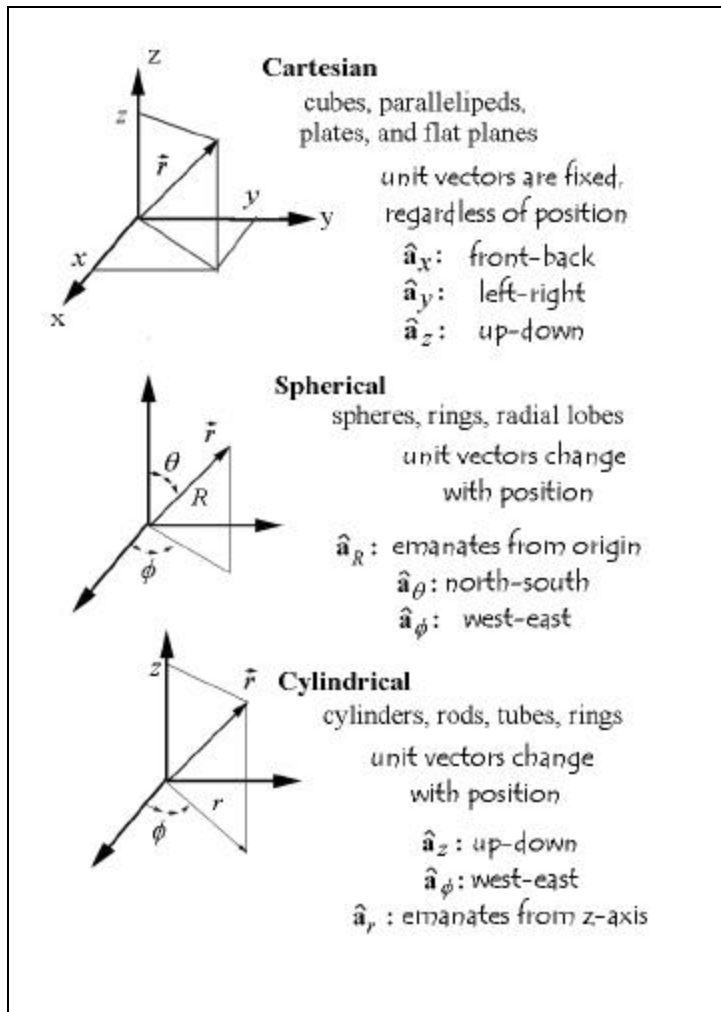
Cross-product: $\vec{A} \times \vec{B} \equiv \vec{a}_n |AB \sin \mathbf{q}_{AB}|$

Useful Relations:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}); \quad \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Orthogonal Coordinate Systems: (In general)

$$\vec{A} = \hat{a}_{u1} A_{u1} + \hat{a}_{u2} A_{u2} + \hat{a}_{u3} A_{u3}$$



line differential element: $d\vec{l} = \hat{a}_{u1}(h_1 du_1) + \hat{a}_{u2}(h_2 du_2) + \hat{a}_{u3}(h_3 du_3)$

surface differential element: $d\vec{s} = \hat{a}_n ds$

$$ds_1 = h_2 h_3 du_2 du_3$$

$$ds_2 = h_1 h_3 du_1 du_3$$

$$ds_3 = h_1 h_2 du_1 du_2$$

volume differential element:

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

Cartesian Coordinates: $u_1 = x, u_2 = y, u_3 = z$; $h_1 = h_2 = h_3 = 1$

$$\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{a}_x (A_y B_z - A_z B_y) + \hat{a}_y (A_z B_x - A_x B_z) + \hat{a}_z (A_x B_y - A_y B_x)$$

$$d\vec{l} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz \text{ (line)}$$

$$ds_x = dydz, ds_y = dx dz, ds_z = dx dy \text{ (area)} \quad dv = dx dy dz \text{ (volume)}$$

Cylindrical Coordinates:

$$u_1 = r, u_2 = \mathbf{f}, u_3 = z ; h_1 = 1, h_2 = r, h_3 = 1 \quad \vec{A} = \hat{a}_r A_r + \hat{a}_f A_f + \hat{a}_z A_z$$

$$\vec{A} \cdot \vec{B} = A_r B_r + A_f B_f + A_z B_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_r & \hat{a}_f & \hat{a}_z \\ A_r & A_f & A_z \\ B_r & B_f & B_z \end{vmatrix} = \hat{a}_r (A_f B_z - A_z B_f) + \hat{a}_f (A_z B_r - A_r B_z) + \hat{a}_z (A_r B_f - A_f B_r)$$

$$d\vec{l} = \hat{a}_r dr + \hat{a}_f r d\mathbf{f} + \hat{a}_z dz \text{ (line element)}$$

$$ds_r = r d\mathbf{f} dz, ds_f = r dr dz, ds_z = r dr d\mathbf{f} \text{ (area element)} \quad dv = r dr d\mathbf{f} dz \text{ (volume element)}$$

Spherical Coordinates:

$$u_1 = R, u_2 = \mathbf{q}, u_3 = \mathbf{f} ; h_1 = 1, h_2 = R, h_3 = R \sin \mathbf{q} \quad \vec{A} = \hat{a}_R A_R + \hat{a}_q A_q + \hat{a}_f A_f$$

$$\vec{A} \cdot \vec{B} = A_R B_R + A_q B_q + A_f B_f$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_R & \hat{a}_q & \hat{a}_f \\ A_R & A_q & A_f \\ B_R & B_q & B_f \end{vmatrix} = \hat{a}_R (A_q B_f - A_f B_q) + \hat{a}_q (A_f B_R - A_R B_f) + \hat{a}_f (A_R B_q - A_q B_R)$$

$$d\vec{l} = \hat{a}_R dR + \hat{a}_q R d\mathbf{q} + \hat{a}_f R \sin \mathbf{q} d\mathbf{f} \text{ (line element)}$$

$$ds_R = R^2 \sin \mathbf{q} d\mathbf{q} d\mathbf{f}, ds_q = R \sin \mathbf{q} R d\mathbf{f}, ds_f = R dR d\mathbf{q} \text{ (area element)}$$

$$dv = R^2 \sin \mathbf{q} dR d\mathbf{q} d\mathbf{f} \text{ (volume element)}$$

Transformations

Cylindrical to Cartesian:

$$x = r \cos \mathbf{f}, \quad y = r \sin \mathbf{f}, \quad z = z$$

$$\hat{a}_x = \hat{a}_r \cos \mathbf{f} - \hat{a}_f \sin \mathbf{f}$$

$$\hat{a}_y = \hat{a}_r \sin \mathbf{f} + \hat{a}_f \cos \mathbf{f}$$

$$\hat{a}_z = \hat{a}_z$$

Cartesian to Cylindrical:

$$r = \sqrt{x^2 + y^2}, \quad \mathbf{f} = \tan^{-1} \frac{y}{x}, \quad z = z$$

$$\hat{a}_r = \hat{a}_x \cos \mathbf{f} + \hat{a}_y \sin \mathbf{f}$$

$$\hat{a}_f = -\hat{a}_x \sin \mathbf{f} + \hat{a}_y \cos \mathbf{f}$$

$$\hat{a}_z = \hat{a}_z$$

Spherical to Cartesian:

$$x = R \sin \mathbf{q} \cos \mathbf{f}, \quad y = R \sin \mathbf{q} \sin \mathbf{f}, \quad z = R \cos \mathbf{q}$$

$$\hat{a}_x = \hat{a}_R \sin \mathbf{q} \cos \mathbf{f} + \hat{a}_q \cos \mathbf{q} \cos \mathbf{f} - \hat{a}_f \sin \mathbf{f}$$

$$\hat{a}_y = \hat{a}_R \sin \mathbf{q} \sin \mathbf{f} + \hat{a}_q \cos \mathbf{q} \sin \mathbf{f} + \hat{a}_f \cos \mathbf{f}$$

$$\hat{a}_z = \hat{a}_R \cos \mathbf{q} - \hat{a}_q \sin \mathbf{q}$$

Cartesian to Spherical:

$$R = \sqrt{x^2 + y^2 + z^2}, \quad \mathbf{q} = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \quad \mathbf{f} = \tan^{-1} \frac{y}{x}$$

$$\hat{a}_R = \hat{a}_x \sin \mathbf{q} \cos \mathbf{f} + \hat{a}_y \sin \mathbf{q} \sin \mathbf{f} + \hat{a}_z \cos \mathbf{q}$$

$$\hat{a}_q = \hat{a}_x \cos \mathbf{q} \cos \mathbf{f} + \hat{a}_y \cos \mathbf{q} \sin \mathbf{f} - \hat{a}_z \sin \mathbf{q} \quad \hat{a}_f = -\hat{a}_x \sin \mathbf{f} + \hat{a}_y \cos \mathbf{f}$$

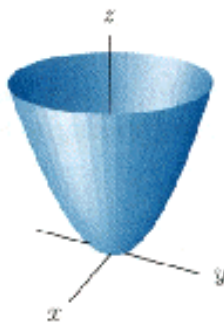


Figure 11.85: Elliptical paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

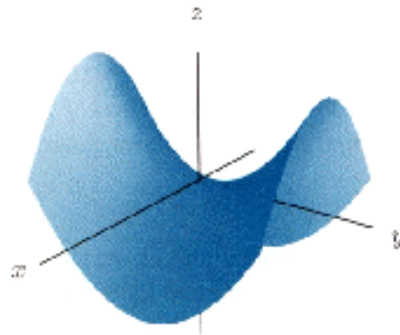


Figure 11.86: Hyperbolic paraboloid $z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$

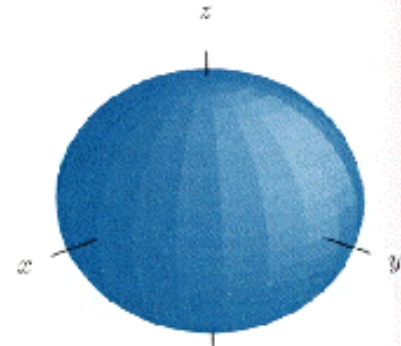


Figure 11.87: Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

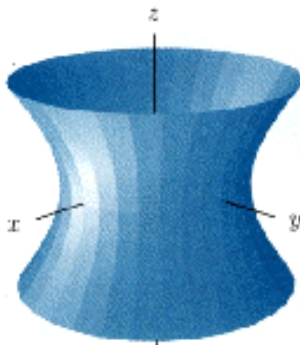


Figure 11.88: Hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

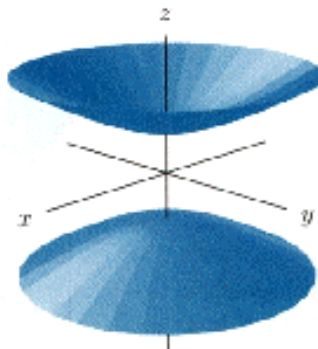


Figure 11.89: Hyperboloid of two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

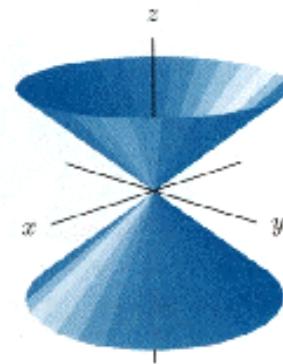


Figure 11.90: Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

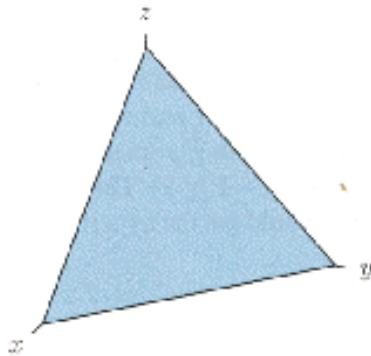


Figure 11.91: Plane $ax + by + cz = d$

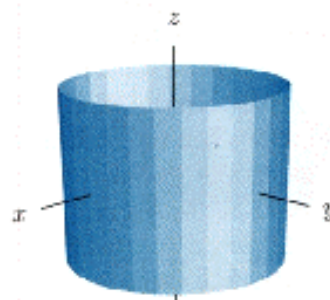


Figure 11.92: Cylindrical surface $x^2 + y^2 = a^2$

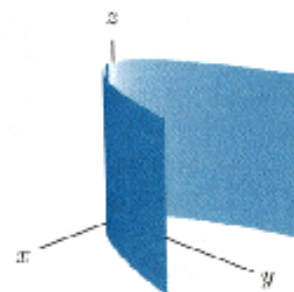


Figure 11.93: Parabolic cylinder $y = ax^2$

VECTOR CALCULUS

Gradient of a Vector (in general):

$$\bar{\nabla}V = \hat{a}_{u_1} \frac{\partial V}{h_1 \partial u_1} + \hat{a}_{u_2} \frac{\partial V}{h_2 \partial u_2} + \hat{a}_{u_3} \frac{\partial V}{h_3 \partial u_3}$$

$$\text{grad } V(u_1, u_2, u_3) \equiv \bar{\nabla}V \equiv \hat{a}_n \frac{dV}{dn}$$

$$dV = (\bar{\nabla}V) \cdot d\mathbf{l}$$

$$\text{Cartesian: } \bar{\nabla}V = \hat{a}_x \frac{\partial V}{\partial x} + \hat{a}_y \frac{\partial V}{\partial y} + \hat{a}_z \frac{\partial V}{\partial z}$$

$$\text{Cylindrical: } \bar{\nabla}V = \hat{a}_r \frac{\partial V}{\partial r} + \hat{a}_f \frac{\partial V}{r \partial \mathbf{f}} + \hat{a}_z \frac{\partial V}{\partial z}$$

$$\text{Spherical: } \bar{\nabla}V = \hat{a}_R \frac{\partial V}{\partial R} + \hat{a}_q \frac{\partial V}{R \partial \mathbf{q}} + \hat{a}_f \frac{\partial V}{R \sin \mathbf{q} \partial \mathbf{f}}$$

Divergence of a Vector:

$$\bar{\nabla} \cdot \bar{A} \equiv \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right]$$

$$\text{Cartesian: } \bar{\nabla} \cdot \bar{A} \equiv \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$$

$$\text{Cylindrical: } \bar{\nabla} \cdot \bar{A} \equiv \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial}{\partial \mathbf{f}} A_f + \frac{\partial}{\partial z} A_z$$

$$\text{Spherical: } \bar{\nabla} \cdot \bar{A} \equiv \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_R) + \frac{1}{R \sin \mathbf{q}} \frac{\partial}{\partial \mathbf{q}} (A_q \sin \mathbf{q}) + \frac{1}{R \sin \mathbf{q}} \frac{\partial A_f}{\partial \mathbf{f}}$$

Curl of a Vector:

$$\bar{\nabla} \times \bar{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{a}_{u_1} h_1 & \hat{a}_{u_2} h_2 & \hat{a}_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$\text{Cartesian: } \bar{\nabla} \times \bar{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\text{Cylindrical: } \bar{\nabla} \times \bar{A} = \frac{1}{r} \begin{vmatrix} \hat{a}_r & \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{a}_r \left(\frac{\partial A_z}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial z} \right) + \hat{a}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{a}_z \left(\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right)$$

Spherical:

$$\bar{\nabla} \times \bar{A} = \frac{1}{R^2 \sin \mathbf{q}} \begin{vmatrix} \hat{a}_R & \hat{a}_\mathbf{q} & \hat{a}_\phi \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \mathbf{q}} & \frac{\partial}{\partial \phi} \\ A_R & RA_\mathbf{q} & R \sin \mathbf{q} A_\phi \end{vmatrix}$$

$$= \frac{\hat{a}_R}{R^2 \sin \mathbf{q}} \left(\frac{\partial(R \sin \mathbf{q} A_\phi)}{\partial \mathbf{q}} - \frac{\partial(RA_\mathbf{q})}{\partial \phi} \right) + \frac{\hat{a}_\mathbf{q}}{R \sin \mathbf{q}} \left(\frac{\partial A_R}{\partial \phi} - \frac{\partial(R \sin \mathbf{q} A_\phi)}{\partial R} \right) + \frac{\hat{a}_\phi}{R} \left(\frac{\partial(RA_\mathbf{q})}{\partial R} - \frac{\partial A_R}{\partial \mathbf{q}} \right)$$

Useful Theorems

$$\text{Stoke's Theorem: } \int_S (\bar{\nabla} \times \bar{A}) \cdot d\bar{s} \equiv \oint_C \bar{A} \cdot d\bar{l}$$

$$\text{Gauss' Theorem: } \int_V (\bar{\nabla} \cdot \bar{A}) dv \equiv \oint_S \bar{A} \cdot d\bar{s}$$

$$\text{Null Identity I: } \bar{\nabla} \times (\bar{\nabla} V) \equiv 0$$

$$\text{Null Identity II: } \bar{\nabla} \cdot (\bar{\nabla} \times \bar{A}) \equiv 0$$

Laplacian operators: $\bar{\nabla}^2 \Phi$

$$\text{Cartesian: } \equiv \frac{\partial^2}{\partial x^2} \Phi + \frac{\partial^2}{\partial y^2} \Phi + \frac{\partial^2}{\partial z^2} \Phi$$

$$\text{Cylindrical: } \equiv \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

$$\text{Spherical: } \equiv \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \Phi}{\partial R} \right) + \frac{1}{R^2 \sin \mathbf{q}} \frac{\partial}{\partial \mathbf{q}} \left(\sin \mathbf{q} \frac{\partial \Phi}{\partial \mathbf{q}} \right) + \frac{1}{R^2 \sin^2 \mathbf{q}} \frac{\partial^2 \Phi}{\partial \phi^2}$$