Cavities

ENEE 381
Role of Cavities

Cavities are resonant structures: Support EM modes at specific frequencies.

Used in:

- Filters
- Oscillators
- Amplifiers
- Measurement of material properties
Resonance

Example:

Transmission line
Terminated in short circuits

\[ \omega = k l \]
\[ k = \frac{\omega}{v} \]
\[ \omega_n = \frac{n \pi l}{v} \quad n = 1, 2, \ldots \]

\[ \frac{d^2}{dx^2} V(x) + \frac{\omega^2}{v^2} V(x) = 0 \]

\[ V(x) = V_+ e^{i\omega x/v} + V_- e^{-i\omega x/v} \]

\[ V(x = 0) = 0 \]
\[ V(x = l) = 0 \]
Enclosed Rectangular Prism

\[ \omega_{nmp} = \sqrt{\left( \frac{n \pi}{a} \right)^2 + \left( \frac{m \pi}{b} \right)^2 + \left( \frac{p \pi}{c} \right)^2} \]

\( n, m, p \) are integers

TE\(_{nm} \) or TM\(_{nm} \)
Modes of a Rectangular WG

\[ E_z = E_0 \sin(k_x x) \sin(k_y y) \]

\[ k_x = \frac{n\pi}{a}, \quad k_y = \frac{m\pi}{b} : \quad n,m = 1, 2, 3, \ldots \]

\[ H_z = H_0 \cos(k_x x) \cos(k_y y) \]

\[ k_x = \frac{n\pi}{a}, \quad k_y = \frac{m\pi}{b} : \quad n,m = 0^*, 1, 2, 3, \ldots \]

* one or the other, but not both

Cut-Off frequencies

\[ \omega_{c,n,m} = v \sqrt{\left( \frac{n\pi}{a} \right)^2 + \left( \frac{m\pi}{b} \right)^2} \]
WG Dispersion Relations

\[ \frac{\omega^2}{v^2} = k_{\perp}^2 + k_z^2 \]

\[ k_z = p \frac{\pi}{L} \]

Cut off frequencies

Transmission Lines Only

Different modes
Enclosed Rectangular Prism

\[ \omega^2 = k_z^2 v^2 + \omega_{c,\text{nm}}^2 \]

\[ \omega_{c,n,m} = \nu\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \]

\[ k_z = p\frac{\pi}{L} \]

\[ \omega_{nmp} = \nu\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{L}\right)^2} \]

TE_{nm} or TM_{nm}
Transmission Line – TEM mode

Operate at frequencies well below TE and TM cut-off
Fabry-Perot Cavity
**Gaussian Beam**

\[
E_{x,y}(x,y,z) = \frac{E_0}{1 + i\frac{z}{Z_R}} \exp\left[-\frac{(x^2 + y^2)}{W_0^2 \left(1 + i\frac{z}{Z_R}\right)} + ikz \right]
\]

\[
W(z) = W_0 \sqrt{1 + \frac{z^2}{Z_R^2}}
\]

Rayleigh Length

\[
Z_R = \frac{1}{2} k W_0^2
\]

\[
E_{x,y}(0,0,z) = \frac{E_0}{1 + i\frac{z}{Z_R}} \exp[ikz] = \frac{E_0}{\sqrt{1 + \left(\frac{z}{Z_R}\right)^2}} \exp[ikz + i\phi(z)]
\]

Guoy Phase \( \tan \phi = -\frac{z}{Z_R} \)
Gaussian Beam

\[ E_{x,y}(x,y,z) = \frac{E_0}{1 + iz / Z_R} \exp \left[ - \frac{(x^2 + y^2)}{W_0^2 (1 + iz / Z_R)} + ikz \right] \]

\[ = \frac{E_0}{\sqrt{1 + z^2 / Z_R^2}} \exp \left[ - \frac{(x^2 + y^2)}{W_0^2 (1 + z^2 / Z_R^2)} \right] \exp \left\{ i \left[ kz + \frac{z(x^2 + y^2)}{Z_R W_0^2 (1 + z^2 / Z_R^2)} + \phi_G \right] \right\} \]

Amplitude

Phase

Pick parameters such that phase is constant on surface of mirror.
The pick k such that the phase changes by \( p\pi \) in going from one mirror to the next.
Wavebeam phase \[ kz + \frac{z(x^2 + y^2)}{Z_RW_0^2 (1 + z^2 / Z_R^2)} + \phi_G \]

Surface of mirror \[ z = \frac{L}{2} - \frac{(x^2 + y^2)}{2R_c} \]

Will match if \[ \frac{L}{2R_c} = \frac{(L/2)^2}{Z_R^2 + (L/2)^2} < 1 \]

Phase change mirror to mirror \[ 2 \left( k_p \frac{L}{2} + \phi_G \right) = p\pi \]

For a given L and \( R_c \), \( Z_R \) is determined above. Hence \( W_0 \) focal spot determined.
Design a Fabrey-Perot resonator

Requirements:
  Wavelength 1 micron = 10^{-6} m.
  Focal spot size = 100 microns = 10^{-4} m.
  Spot size on mirrors = 300 microns = 3\times10^{-4} m

Find L and R_c

Super bonus: How big must the mirrors be to keep “spill over” below 10%
Quality Factor

\[
\frac{1}{Q} \equiv \frac{\text{Power Dissipated}}{\text{Energy Stored}} \\
\omega \text{ Energy Stored} \\
\text{Field decay rate} \\
E, H \sim \exp \left( -\frac{\omega}{2Q} t \right)
\]
Frequency Domain

Steady state field response

$E, H \sim \frac{\text{Source}}{\omega - \omega_{\text{res}} + i \frac{\omega_{\text{res}}}{2Q}}$

$|E|^2, |H|^2 \sim \frac{1}{\left( \omega - \omega_{\text{res}} \right)^2 + \left( \frac{\omega_{\text{res}}}{2Q} \right)^2}$

Half maximum $\omega - \omega_{\text{res}} = \pm \frac{\omega_{\text{res}}}{2Q}$

Full Width at Half Maximum (FWHM) = $\frac{\omega_{\text{res}}}{Q}$
Response Function

\[ |E|^2 \propto \frac{1}{(w-w_{\text{res}})^2 + \left(\frac{w_{\text{res}}}{2Q}\right)^2} \]

When \( w = w_{\text{res}} \pm \frac{w_{\text{res}}}{2Q} \)

\[ |E|^2 \text{ is } \frac{1}{2} \text{ of peak value} \]

Full width at half max of Power (FWHM) \( \frac{w_{\text{res}}}{Q} \)
Multiple Contributions to Loss

\[
\frac{1}{Q} = \frac{P_d}{\omega U} = \frac{P_{d1}}{\omega U} + \frac{P_{d2}}{\omega U} + \cdots
\]

if losses are small \((\omega \gg 1)\)

losses are additive

\[
\frac{1}{Q} = \frac{1}{Q_1} + \frac{1}{Q_2} + \cdots
\]

Reciprocals of \(Q\) add.
Dielectric and Conductor Loss

\[ Q \text{ due to lossy dielectric} \]

\[ \varepsilon = \varepsilon' - j\varepsilon'' \]

\[ Q = \frac{\varepsilon'}{\varepsilon''} \]

\[ \varepsilon \text{ must completely fill cavity} \]

Losses due to conductors

\[ Q = \frac{\omega}{R_s} \left( \frac{\mu}{\mu'} \right) \frac{\int d^3 x \mid \mathbf{H} \mid^2}{\int d^a \mid \mathbf{H}_t \mid^2} \rightarrow \text{energy stored} \]

\[ \rightarrow \text{losses} \]
Coupling to Cavities

A closed box is useless.

Need to be able to get power in it.

Adding a hole (or coupling port) does two things: power can come in and power can go out.
Coupling Also Characterized by $Q$
Critically Coupled

\[ Q_{\text{coupling}} = Q_{\text{internal}} \]

At resonance, all power from source is absorbed in cavity.

\[ V_{\text{incident}} \]
\[ V_{\text{reflected}} \]

Voltage reflection coefficient

\[ \rho = \frac{V_{\text{reflected}}}{V_{\text{incident}}} \]

\[ \rho = - \frac{i \left( \frac{\omega}{\omega_{\text{res}} - 1} \right) - \left( \frac{1}{2Q_i} - \frac{1}{2Q_e} \right)}{i \left( \frac{\omega}{\omega_{\text{res}} - 1} \right) - \left( \frac{1}{2Q_i} + \frac{1}{2Q_e} \right)} \]
Universal Response

\[
\rho = -\frac{\left(\frac{i}{w_{\text{res}} - 1} - \left(\frac{1}{2Q_i} - \frac{1}{2Q_e}\right)\right)}{\left(\frac{1}{2Q_i} + \frac{1}{2Q_e}\right)}
\]

Knowing \(Q_i\), \(Q_e\) and \(w_{\text{res}}\) determines \(\rho\).

Far from resonance \(\rho = -1\).

Reflectivity at resonance

\[
|P|_{\text{res}}^2 = \left|\frac{Q_i^{-1} - Q_e^{-1}}{Q_i^{-1} + Q_e^{-1}}\right|^2
\]

0 - if \(Q_i = Q_e\)

Damping rate for fields \(\rho \to \infty\)

\[
\frac{w}{w_{\text{res}}} = 1 - \frac{i}{2Q_e}
\]

\[
\frac{1}{Q_i} = \frac{1}{Q_0} + \frac{1}{Q_e}
\]
Waveguide Cavities

CAVITIES constructed from cylindrical waveguides will have normal modes

\[ z = 0 \quad z = L \]

metal walls \[ \frac{\partial E_z}{\partial z} = 0 \]
Waveguide fields

\[ \mathbf{E} = \text{Re}\left\{ \hat{\mathbf{E}}(x, y) \exp\left[ i \left( k_z z - \omega t \right) \right] \right\} \]

\[ \mathbf{H} = \text{Re}\left\{ \hat{\mathbf{H}}(x, y) \exp\left[ i \left( k_z z - \omega t \right) \right] \right\} \]

\[ \frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} = -\left( \frac{\omega}{v} \right)^2 k_z^2 \hat{H}_z \]

\[ \frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} = -\left( \frac{\omega}{v} \right)^2 k_z^2 \hat{E}_z \]

\[ \hat{E}_\perp = \frac{i}{(\omega / v)^2 - k_z^2} \left[ k_z \nabla \perp \hat{E}_z - \omega \mu \hat{\mathbf{z}} \times \nabla \perp \hat{H}_z \right] \]

\[ \hat{H}_\perp = \frac{i}{(\omega / v)^2 - k_z^2} \left[ k_z \nabla \perp \hat{H}_z + \omega \varepsilon \hat{\mathbf{z}} \times \nabla \perp \hat{E}_z \right] \]

\[ \left. \hat{E}_z \right|_{\text{wall}} = 0, \quad \left. \mathbf{n} \cdot \nabla \perp \hat{H}_z \right|_{\text{wall}} = 0 \]
Forward and Backward Waves

TM modes

\[ E_z = \hat{E}_{z,nm}(x, y)(A_+ \exp(ik_z z) + A_- \exp(-ik_z z)) \]

\[ \hat{E}_\perp = \frac{ik_z \nabla \perp \hat{E}_{z,nm}}{\left( \omega / v \right)^2 - k_z^2} \left( A_+ \exp(ik_z z) - A_- \exp(-ik_z z) \right) \]

BC: \( \hat{E}_\perp(z=0,L)=0 \)

\( A_+ = A_- \quad A_+ = A_- e^{-2ik_z L} \)

TE modes

\[ H_z = H_{z,nm}(x, y)(A_+ \exp(ik_z z) + A_- \exp(-ik_z z)) \]

\[ \hat{E}_\perp = \frac{-i\omega \mu \hat{\mathbf{z}} \times \nabla \perp \hat{H}_{z,nm}}{\left( \omega / v \right)^2 - k_z^2} \left( A_+ \exp(ik_z z) + A_- \exp(-ik_z z) \right) \]

k_z L = p\pi, \quad p = 0, 1, 2, ...
Resonant Frequencies

\[ k_{ii} = \frac{\pi p}{L} \]

\[ \omega^2 \varepsilon \mu = k_e^2 + \left( \frac{\pi p}{L} \right)^2 \]

determined by cross section

\[ \omega_{\text{Res}} = \frac{c}{\sqrt{\varepsilon \mu}} \sqrt{k_e^2 + \left( \frac{\pi p}{L} \right)^2} \]

Rectangular cross section

\[ \omega_{\text{Res}} = \omega_{\text{Uni}} = \frac{c}{\sqrt{\varepsilon \mu}} \sqrt{\left( \frac{\pi m}{a} \right)^2 + \left( \frac{\pi n}{b} \right)^2 + \left( \frac{\pi p}{L} \right)^2} \]
TM Modes

\[ E_\parallel = \frac{1}{2} \left\{ \hat{E}_n(x_\perp) \left[ A_+ e^{i(k_n z - \omega t)} + A_- e^{-i(k_n z - \omega t)} \right] + \text{c.c.} \right\} \]

\[ E_\perp = \frac{1}{2} \left\{ \frac{i k_n \nabla_{\perp} \hat{E}_n}{k_c^2} \left[ A_+ e^{i(k_n z - \omega t)} - A_- e^{-i(k_n z - \omega t)} \right] \right\} \]

\[ a \perp \quad g = 0 \quad A_+ - A_- = 0 \quad A_+ = A_- \]

\[ a \perp \quad 3 = \phi \quad A_+ e^{ik_n L} = A_- e^{-ik_n L} \]

again

\[ 2k_n L = 2\pi p \quad p = 0 \quad \text{O.K.} \]
Cavity Losses

\[ Q = \frac{\omega U}{P_d} \quad \text{energy stored} \]
\[ P_d \quad \text{power dissipated} \]

**Quality Factor**

**Poynting Theorem**

\[ \frac{\partial}{\partial t} \int d^3 x \frac{1}{2} (\varepsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2) + \int dA \cdot \frac{1}{2} \text{Re} \{ \mathbf{E} \cdot \mathbf{H}^* \} = 0 \]

For cavity modes

\[ \int d^3 x \frac{\varepsilon |\mathbf{E}|^2}{4} = \int d^3 x \frac{\mu |\mathbf{H}|^2}{4} \quad \text{average energy stored is equal} \]

\[ P_d = \int dA \left( \frac{1}{2} \varepsilon |\mathbf{E}|^2 \right) \]

\[ Q = \frac{\omega}{c} \sqrt{\frac{1}{\varepsilon R_3} \frac{\int d^3 x |\mathbf{H}|^2}{\int dA |\mathbf{H}^*|^2}} \]
Weyl’s Formula

How many modes in a cavity of volume \( V \) have \( \omega \) = \( \omega_0 \)?

Consider a cubic cavity of side \( L \)  

\[ V = L^3 \]

Resonant Frequencies

\[ \omega_n = \frac{\pi c}{L} \sqrt{n_1^2 + n_2^2 + n_3^2} \]

where

\[ n = (n_1, n_2, n_3) \]
Estimate of Number of Modes

How many combinations of integers $(n_x, n_y, n_z)$ have

\[ n_x^2 + n_y^2 + n_z^2 < \left( \frac{\omega c}{n} \right)^2 \left( \frac{kL}{\pi} \right)^2 \]

\[ \omega_n = \frac{\pi c}{L} \sqrt{n_x^2 + n_y^2 + n_z^2} \]

\[ n = (n_x, n_y, n_z) \]

Every combination occupies a cube of unit volume.
Volume Inside Spherical Surface

\[ N = \frac{1}{18} \frac{4\pi}{3} \left( \frac{kL}{\pi} \right)^3 \]

\[ N(k) = \frac{(kL)^3}{6\pi^2} = \frac{k^3V}{6\pi^2} \]

But wait! For each set of integers there are 2 polarizations.

For EM modes \[ N(k) = \frac{1}{3} \frac{k^3V}{\pi^2} \]
Example

\[ \text{Volume} = 7 \text{ m}^3 \]

\[ f = 1 \text{ GHz} \]

\[ \frac{k}{c} = \frac{2\pi f}{3 \times 10^8} = 2.1 \text{ m}^{-1} \]

\[ N(k) = 310 \]

What is the typical spacing?

\[ \delta k = \text{spacing in } k = \omega / c \]

\[ N(k + \delta k) = N(k) + 1 \]

\[ k^2 V \]

\[ \frac{\delta k}{k} = \frac{1}{k \frac{dN}{dk}} = \frac{\pi^2}{k^3 V} \approx 1.07 \times 10^{-3} \]
Integrated photonics
(Courtesy Edo Waks)
Klystron – Beam Driven HPM source

**Klystron**: invented in 1937 by the Varian brothers. One of the first Palo Alto High Tech. firms.  
**High Power Source of Microwaves**  
Radar, Particle Accelerators, (LHC 16 x 300 kW), etc
Field in cavity 1 gives small time dependent velocity modulation.

Fast electrons catch up to slow electrons giving large current modulation.
Examples

170 GHz CPI Gyrotron
IEEE IVEC
http://ieeexplore.ieee.org

Experimental high power set-up showing the CPI 218.4 GHz EIK driving the compact NRL Serpentine Waveguide (SWG) TWT.

L3 Ka Band Power Module
http://www.linkmicrotek.com

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