Do the following problems from the text

P5.1, P5.3, P5.9, P5.10, P5.22

Also

You are going to determine the resistance per unit length of a planar block of conducting material that has two contacts as shown below. Assume the block extends infinitely in the z direction, which is out of the page.

The lengths L, W, and T, as well as the conductivity are variable parameters.

As a first step you will need to formulate the problem and determine which equations you will solve numerically and what boundary conditions you will apply. Also you will need to decide what information from the solution, \( V(x,y) \) is necessary to determine the resistance per unit length.
Chapter 5

Steady Electric Currents

P. 5-1 a) Integrating Eq. (5-16): $V(y) = \left[ \frac{9\pi}{4\varepsilon_0 \sqrt{2e}} \right]^{3/2} \frac{y^{3/2}}{y_0^{3/2}}$

$E(y) = -\frac{\sigma}{y} \frac{dV(y)}{dy} = -\frac{\sigma}{y} \frac{4V_0}{3d} \left( \frac{y}{y_0} \right)^{1/2}$.

b) $\sigma(y) = \frac{\varepsilon_0}{\sigma} \frac{dE(y)}{dy} = -\frac{4\varepsilon_0 V_0}{9d^2} \left( \frac{y}{d} \right)^{3/2}$

$Q = \int_0^d \sigma(y) \frac{d^2}{dy} dy = -\frac{4\varepsilon_0 V_0 s}{3d} \int_0^d y^{-1/2} dy = -\frac{4\varepsilon_0 V_0 s}{3d}$.

c) On the anode, $y = d$, $\sigma = -\sigma_0 E(d) = -\frac{4\varepsilon_0 V_0}{3d}$.

Total surface charge on anode = $\sigma_0 s = \frac{4\varepsilon_0 V_0}{3d} s$.

Total charge on cathode = 0.

d) Substituting 1 in Eq. (5-12):

$u = \frac{dV}{dt} = \left( \frac{2eV_0}{m} \right)^{1/2} \left( \frac{y}{d} \right)^{3/2}$,

$y^{-1/2} dy = \frac{(2eV_0)^{1/2}}{m \sigma_0 s/3} dt$.

Integrating: $y = \left( \frac{2eV_0}{9m \sigma_0 s/3} \right)^{3/2} t^2$.

\[ y = d \]

"Transit time $T_t = 3d \left( \frac{m}{2eV_0} \right)^{1/2}$."

For $V_0 = 200 \text{ (V)}$, $d = 10^{-2} \text{ (m)}$, $m = 9.1 \times 10^{-31} \text{ (kg)}$, and $e = 1.6 \times 10^{-19} \text{ (C)}$, $T_t = 3.58 \times 10^{-9} \text{ (s)} = 3.58 \text{ (ns)}$.

P. 5-3 $R_1$ = Resistance per unit length of core $= \frac{1}{\sigma_0 s_1} = \frac{1}{\sigma_0 \pi a^2}$.

$R_2$ = Resistance per unit length of coating $= \frac{1}{\sigma_0 s_2}$.

Let $b$ = Thickness of coating, $S_2 = \pi(a+b)^2 - \pi a^2 = \pi(2ab+b)$.

a) $R_1 = R_2 \rightarrow b = (\sqrt{11} - 1)a = 2.32a$.

b) $I_1 = I_2 = \frac{1}{2}$.

$J_1 = \frac{I_1}{2\pi a^2}$, $J_2 = \frac{I_2}{2s_2}$

$E_1 = \frac{J_1}{\sigma} = \frac{1}{2\pi a^2}$, $E_2 = \frac{J_2}{\sigma s_2} = \frac{1}{2\pi a^2}$.

Thus, $J_1 = 10J_2$ and $E_1 = E_2$. 
Homework #7 Solutions

Problem 5.9
This problem makes use of boundary conditions that we have learned thus far...

a)
\[ E_{1t} = E_{2t} \]
\[ E_2 \sin \alpha_2 = E_1 \sin \alpha_1 \]
\[ J_{1n} = J_{2n} \]
\[ \sigma_2 E_2 \cos \alpha_2 = \sigma_1 E_1 \cos \alpha_1 \]

Solving these two equations yields the magnitude and angle of \( E_2 \)

\[ E_2 = E_1 \sqrt{\sin^2 \alpha_1 + \left( \frac{\sigma_1}{\sigma_2 \cos \alpha_1} \right)^2} \]
\[ \tan \alpha_2 = \frac{\sigma_2}{\sigma_1} \tan \alpha_1 \]

b)
\[ D_{2n} - D_{1n} = \rho_s \]
\[ \varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \rho_s \]
\[ \rho_s = \left( \frac{\sigma_1}{\sigma_2} \varepsilon_2 - \varepsilon_1 \right) E_1 \cos \alpha_1 \]

c)
If both media are perfect dielectrics, then \( \sigma_1 = \sigma_2 = 0 \), then \( \rho_s = 0 \) and the magnitude and angle of the electric field will be given by equations (3-129) and (3-130) on page 119.

Problem 5.10

\[ \sigma(y) = \sigma_1 + (\sigma_2 - \sigma_1)(y/d) \]

a)
Neglecting fringing effects and assuming a current density...

\[ J = -a_y J_0 \]
\[ E = J/\sigma = -a_y J_0/\sigma(y) \]
\[ V_0 = \int_0^d E \, dy = \int_0^d J_0 / \{ \sigma_1 + (\sigma_2 - \sigma_1)(y/d) \} \, dy = J_0 d / (\sigma_2 - \sigma_1) \ln(\sigma_2/\sigma_1) \]
\[ R = V_0 / I = V_0 / (JS) = \left( \frac{d}{S(\sigma_2 - \sigma_1)} \right) \ln \left( \frac{\sigma_2}{\sigma_1} \right) \]

b)
\[ \rho_s,upper = \varepsilon_0 E_0(d) = \varepsilon_0 J_0 / \sigma_2 = \varepsilon_0 \left( \frac{\sigma_2 - \sigma_1}{\sigma_1} \right) V_0 / \{ \sigma_2 d \ln \left( \frac{\sigma_2}{\sigma_1} \right) \} \]
\[ \rho_s,lower = -\varepsilon_0 E_0(0) = -\varepsilon_0 J_0 / \sigma_1 = \varepsilon_0 \left( \frac{\sigma_2 - \sigma_1}{\sigma_2} \right) V_0 / \{ \sigma_1 d \ln \left( \frac{\sigma_2}{\sigma_1} \right) \} \]

c)
\[ \rho = \nabla D = d/dy(\varepsilon_0 E) = -\varepsilon_0 J_0 d / d'y(1 / \{ \sigma_1 + (\sigma_2 - \sigma_1)(y/d) \}) = \{ \varepsilon_0 J_0 (\sigma_2 - \sigma_1) / d \} / \{ \sigma_1 + (\sigma_2 - \sigma_1)(y/d) \} \]
Problem 5.22
The specified boundary conditions can be satisfied by solutions of Laplace's equation with zero separation constants: \( k_x = k_y = 0 \) (See table 4.1)

\[
\begin{align*}
X(x) &= A_0x + B_0 \\
Y(y) &= C_0y + D_0
\end{align*}
\]

From first boundary condition, \( B_0 = C_0 = 0 \) ...

\[
V(x) = A_0D_0x
\]

a) Applying the other boundary conditions ...

\[
\begin{align*}
V(a) &= V_0 = A_0D_0a \\
V &= (V_0/a)x
\end{align*}
\]

b) \[
\begin{align*}
E &= -\nabla V = -aV_0/a \\
J &= \sigma E = -a\sigma V_0/a
\end{align*}
\]

Extra Comment
For more examples of the successive over relaxation algorithm, consult the Electrostatics with MATLAB course supplement. In it there is a program called the Laplacian Solver which solves numerically solves rectangular boundary value problems.